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1997 J. Phys. A: Math. Gen. 30 2027

(http://iopscience.iop.org/0305-4470/30/6/025)

J. Phys. A: Math. Gen. **30** (1997) 2027–2054. Printed in the UK PII: S0305-4470(97)75727-5

Quantum principal bundles and corresponding gauge theories

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Received 18 June 1996

Abstract. A generalization of the classical gauge theory is presented, in which compact quantum groups play the role of the internal symmetry groups. All considerations are performed in the framework of a noncommutative-geometric formalism of locally trivial quantum principal bundles over classical smooth manifolds. Quantum counterparts of classical gauge bundles, and classical gauge transformations, are introduced and investigated. A natural differential calculus on quantum gauge bundles is constructed and analysed. Kinematical and dynamical properties of corresponding gauge theories are discussed. Particular attention is given to the purely quantum phenomena appearing in the formalism, and their physical interpretation. An example with quantum $SU(2)$ group is considered.

1. Introduction

The gauge field theory is one of the fundamental theoretical tools for the development of unifying models of elementary particles and their interactions. The basic idea incorporated in the gauge formalism is that of local symmetry, so that the internal symmetry transformations can be performed independently in various spacetime points. A consistent formulation of such a theory is possible by introducing fields of a special nature—gauge fields. These fields appropriately 'compensate' effects of arbitrary local symmetry transformations of the standard matter fields. The group of local tranformations is infinite-dimensional. In particular, as a consequence of this high-level symmetry, gauge theory is always renormalizable [23].

The simplest example of a gauge theory is given by classical and quantum electrodynamics. In this case the internal symmetry group is simply $U(1)$. Electrodynamics is included in a non-Abelian gauge theory unifying electromagnetic and weak interactions. The internal symmetry group for electroweak interactions is given by $U(1) \times SU(2)$. Furthermore, the physics of quarks and strong interactions is, at least phenomenologically, includable into the same conceptual framework. Finally, the general relativity theory can be viewed as a special case of gauge theory, associated to the Poincare group.

On the other hand, gauge theory is a paradigmic example of the interplay between fundamental physics and differential geometry [7, 16]. The appropriate differential geometric framework is given by principal bundles [18, 19]. The physical spacetime plays the role of the base manifold of the bundle, and the structure group is identified with the group describing internal symmetries. Gauge fields are interpreted as connection forms, while the matter fields are sections of the appropriate associated vector bundles. The infinite-dimensional group of gauge transformations is identified with the group of vertical automorphisms of the bundle.

If the gauge theory really reflects something fundamental in nature, then it is plausible to assume that the basic picture holds also at the level of ultra-small distances (the Planck scale). However, the classical smooth manifold description of spacetime is not appropriate at this level.

Such a natural way of thinking leads to the assumption that general philosophy of gauge theory should be independent of the nature of the underlying spacetime.

Noncommutative differential geometry [5, 6] essentially enlarges the classical idea of geometric space, introducing the concept of a quantum space. Informally speaking, a quantum space shows ‘quantum fluctuations’ of geometry and it is not understandable in the standard way—as a collection of points equipped with an additional structure. Quantum spaces are described by the appropriate non-commutative *-algebras, the elements of which are interpretable as ‘smooth functions’ over these spaces.

There exist non-trivial reasons [6] to believe that noncommutative geometry is capable to provide the appropriate description of the spacetime at the Planck scale, and to overcome deep mathematical inconsistencies present in the standard formulation of quantum field theory.

In conjunction with the generalization of the concept of space, noncommutative geometry opens a possibility of extending the concept of symmetry—via quantum groups. Geometrically, quantum groups are quantum spaces endowed with a group structure and they are included within the framework of Hopf algebras [1].

It is therefore natural to look for the appropriate noncommutative-geometric generalization of standard gauge theory. A natural mathematical framework for such a theory should be given by *quantum principal bundles*, where quantum groups play the role of structure groups and quantum spaces are the counterparts of base manifolds. Such a general formalism was built in papers [8–10]. A short presentation is given in [11]. Conceptually similar, but technically different a formulation of quantum principal bundles was presented in [2].

In this study we shall discuss properties of a gauge theory based on quantum principal bundles. We shall assume here that the spacetime is described by classical geometry, and use locally trivial quantum principal bundles over classical manifolds [8] accordingly. However, general compact matrix quantum groups [26] will play the role of entities describing local symmetries.

This paper is a first step towards a ‘fully quantum’ gauge theory [14] in which the spacetime will also be quantum. There exists several reasons why it is plausible to consider separately the special case of the theory over a classical spacetime.

First, it is technically easier to deal with classical base manifolds, where the concept of locality is clearly defined and computations can be performed in local trivializations, as in the standard theory. In particular, the formalism gives an alternative (but equivalent) way to perform calculations in the standard gauge theory (and surprisingly, such calculations are simpler than the conventional ones). In general quantum context, it is not possible to introduce the concept of a local trivialization, and all considerations should be performed ‘globally’.

On the other hand, it turns out that various global components of the fully quantum formalism of quantum principal bundles [9] are essentially the same as in the special theory [8], being independent of the nature of the base manifold.

Finally, in noncommutative geometry the concept of space and the concept of symmetry are completely independent. Therefore it would be interesting to see what the separate contribution of the idea of local quantum group symmetry is to the structure of the corresponding gauge theory. As we shall see, introducing quantum groups leads to various 'purely quantum' phenomena. The enlarged concept of symmetry opens, in principle, a possibility of further unifying standard particle multiplets into quantum group 'supermultiplets' and it is natural to expect that the theory possesses the same amiable properties as its classical-geometric counterpart.

Let us outline the contents of this paper. In the next section, preparatory material is collected. First, we fix the notation and introduce relevant quantum group entities. Secondly, we present the most important ideas and results of [8], which will be used in the main considerations. As we have mentioned, the starting point for all constructions will be a quantum principal G-bundle, P, over a smooth manifold M, playing the role of spacetime, while G is a compact matrix quantum (structure) group [26] representing 'local symmetries' of the system.

In section 3 a quantum analogue of the gauge bundle will be constructed and investigated. This quantum bundle (over M) will be denoted by $\mathcal{G}(P)$. Various quantum counterparts of gauge transformations are naturally associated to $\mathcal{G}(P)$. Further, a differential calculus on the bundle $\mathcal{G}(P)$ will be constructed, by combining the standard differential calculus on M (based on differential forms) with an appropriate differential calculus on the quantum group G. This calculus on $\mathcal{G}(P)$ is relevant in situations in which quantum counterparts of gauge transformations act on entities related to differential calculus on the principal bundle, P, as connection forms for example.

It is important to mention that there exist two natural inequivalent ways of introducing quantum counterparts of gauge transformations. The first one is to translate into the quantum context the idea that gauge transformations are vertical automorphisms of the principal bundle P.

This approach leads to a standard group (of gauge transformations of P). The same group will be obtained if we consider counterparts of sections of the bundle $\mathcal{G}(P)$. However, it turns out that such a concept of a gauge transformation does not describe gauge-like phenomena related to the quantum nature of the space G. Namely, because of the inherent geometrical inhomogeneity of quantum groups, every quantum principal bundle, P, over M is completely determined by its classical part, P_{cl} (interpretable as the set of points of P). The classical part is an ordinary principal G_{cl}-bundle over M, where G_{cl} is a group (the classical part of G) interpretable as consisting of points of G. We shall prove that gauge transformations of P are in a natural bijection with standard gauge transformations of P_{cl}. Further, we shall prove that the set of points of $\mathcal{G}(P)$ coincides, in a natural manner, with the standard gauge bundle $\mathcal{G}(P_{\mathrm{cl}})$.

The second approach to gauge transformations is in some sense indirect. The main idea is to construct the 'action' of the bundle $\mathcal{G}(P)$ on P (generalizing the classical situation). This approach does not meet geometrical obstructions. In classical geometry, the mentioned action naturally contains all the information about gauge transformations.

Section 4 is devoted to the formulation and kinematical and dynamical analysis of quantum group gauge theories, in the framework of quantum principal bundles. Gauge fields will be geometrically represented by connections on P. Internal degrees of freedom of such gauge fields are determined by fixing a bicovariant first-order differential *-calculus [27] on the structure quantum group G. In this paper we shall deal with a unique differential calculus on G which can be characterized as the minimal bicovariant differential calculus compatible, in appropriate sense, with the geometrical structure on the bundle P. If we start from this calculus on the group then it is possible to built natural differential calculi on bundles P and $\mathcal{G}(P)$ which are always 'locally trivialized' when bundles P (and therefore

$\mathcal{G}(P)$) are locally trivialized.

Dynamical properties of the gauge theory will be determined after fixing an appropriate Lagrangian. In analogy with the classical gauge theory, we shall consider Lagrangians which are quadratic functions of the curvature form. We shall compute the corresponding equations of motion. Symmetry properties of the introduced Lagrangian will be analysed. We shall prove the invariance of the Lagrangian under the action of the (ordinary) group of gauge transformations of P. Further, it turns out that the Lagrangian is invariant, in an appropriate sense, under the natural action of $\mathcal{G}(P)$ on P. This corresponds to the full gauge invariance of the Lagrangian in the classical theory.

In section 5 everything will be illustrated in a conceptually simple but technically highly non-trivial example in which G is the quantum $SU(2)$ group. The most important observation is that the corresponding gauge theory is *essentially different* from the classical $SU(2)$ gauge theory, and does not reduce to the classical theory when the deformation parameter $1-\mu$ tends to zero. This is caused by the fact that the minimal admissible bicovariant calculus does not respect the classical limit. Namely, a detailed analysis [8] shows that for $\mu \in (-1, 1) \setminus \{0\}$ the space of left-invariant elements (playing the role of the dual space of the corresponding Lie algebra) of the mentioned minimal calculus is infinite-dimensional, and can be naturally identified with the algebra of polynomial functions on a quantum two-sphere [21]. Hence, the corresponding gauge fields possess infinitely many internal degrees of freedom, in contrast to the classical case. Finally, in section 6 concluding remarks are made.

The paper is brought to a close with an appendix, in which some technical properties related to the minimal admissible bicovariant calculus on the quantum $SU(2)$ group are collected.

2. Mathematical background

Let G be a compact matrix quantum group [26]. We shall denote by $\mathcal{A}$ the *-algebra of 'polynomial functions' on G, and by $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\epsilon : \mathcal{A} \to \mathbb{C}$ and $\kappa : \mathcal{A} \to \mathcal{A}$ the coproduct, co-unit and the antipode, respectively. The symbols $a^{(1)} \otimes \cdots \otimes a^{(n)}$ will be used for the result of an $(n-1)$-fold coproduct of an element $a \in \mathcal{A}$ (so that $\phi(a) = a^{(1)} \otimes a^{(2)}$). Let G_{cl} be the classical part [8] of G. Explicitly, G_{cl} consists of *-characters (nontrivial multiplicative linear Hermitian functionals) of $\mathcal{A}$. The Hopf algebra structure on $\mathcal{A}$ naturally induces the group structure on G_{cl}, such that

$$gg' = (g \otimes g')\phi$$
$$g^{-1} = g\kappa$$

for each $g, g' \in G_{\mathrm{cl}}$. The co-unit $\epsilon : \mathcal{A} \to \mathbb{C}$ is the neutral element of G_{cl}. We shall assume that the (complex) Lie algebra $\mathrm{lie}(G_{\mathrm{cl}})$ is realized [8] as the space of linear functionals $X : \mathcal{A} \to \mathbb{C}$ satisfying

$$X(ab) = \epsilon(a)X(b) + \epsilon(b)X(a)$$

for each $a, b \in \mathcal{A}$.

Let Γ be a first-order differential calculus over G. This means [27] that Γ is a bimodule over $\mathcal{A}$ endowed with a differential $\mathrm{d} : \mathcal{A} \to \Gamma$ such that elements of the form $a\,\mathrm{d}b$ linearly generate Γ. Let

$$\Gamma^{\otimes} = \sum_{k \geqslant 0}^{\oplus} \Gamma^{\otimes k}$$

be the tensor bundle algebra [27] built over Γ. Let

$$\Gamma^\wedge = \sum_{k\geqslant 0}^{\oplus} \Gamma^{\wedge k}$$

be the universal differential envelope ([8, appendix B]) of Γ. The algebra $\Gamma^\wedge$ can be obtained from $\Gamma^\otimes$ by factorizing through the ideal $S^\wedge \subseteq \Gamma^\otimes$ generated by the elements of the form

$$Q = \sum_i \mathrm{d}a_i \otimes_{\mathcal{A}} \mathrm{d}b_i$$

where $a_i, b_i \in \mathcal{A}$ satisfy $\sum_i a_i \mathrm{d}b_i = 0$. In particular, the differential $\mathrm{d} : \Gamma^\wedge \to \Gamma^\wedge$ extends $\mathrm{d} : \mathcal{A} \to \Gamma$, in a natural manner.

Let us assume that Γ is left-covariant [27] and let $\ell_\Gamma : \Gamma \to \mathcal{A} \otimes \Gamma$ be the left action of G on Γ. Let Γ_{inv} be the space of left-invariant elements of Γ (playing the role of the dual space of the Lie algebra of G) and let $\pi : \mathcal{A} \to \Gamma_{\mathrm{inv}}$ be the canonical projection map, given by

$$\pi(a) = \kappa(a^{(1)})\,\mathrm{d}a^{(2)}.$$

This map is surjective and $\mathcal{R} = \ker(\epsilon) \cap \ker(\pi)$ is the right $\mathcal{A}$-ideal which canonically [27] corresponds to Γ.

The space Γ_{inv} possesses a natural right $\mathcal{A}$-module structure, which will be denoted by $\circ$. Explicitly

$$\pi(a) \circ b = \pi[(a - \epsilon(a)1)b]$$

for each $a, b \in \mathcal{A}$.

Let us now assume that Γ is bicovariant, and let $\wp_\Gamma : \Gamma \to \Gamma \otimes \mathcal{A}$ be the right action of G on Γ.

The 'adjoint' action $\mathrm{ad} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of G on G is given by

$$\mathrm{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}.$$

The space Γ_{inv} is right-invariant, that is $\wp_\Gamma(\Gamma_{\mathrm{inv}}) \subseteq \Gamma_{\mathrm{inv}} \otimes \mathcal{A}$. The corresponding restriction $\varpi : \Gamma_{\mathrm{inv}} \to \Gamma_{\mathrm{inv}} \otimes \mathcal{A}$ is interpretable as the adjoint action of G on Γ_{inv}. Explicitly ϖ is characterized by

$$\varpi\pi = (\pi \otimes \mathrm{id})\,\mathrm{ad}\,.$$

The actions ℓ_Γ and $\wp_\Gamma$ can be naturally extended to the grade preserving homomorphisms $\wp_\Gamma^{\wedge,\otimes} : \Gamma^{\wedge,\otimes} \to \Gamma^{\wedge,\otimes} \otimes \mathcal{A}$ and $\ell_\Gamma^{\wedge,\otimes} : \Gamma^{\wedge,\otimes} \to \mathcal{A} \otimes \Gamma^{\wedge,\otimes}$ (their restrictions on $\mathcal{A}$ coincide with ϕ).

The symbol $\hat{\otimes}$ will be used for the graded tensor product of graded-differential algebras. The coproduct, ϕ, admits the unique extension $\hat{\phi} : \Gamma^\wedge \to \Gamma^\wedge \hat{\otimes} \Gamma^\wedge$ which is a homomorphism of graded-differential algebras [8]. In particular,

$$\hat{\phi}(\xi) = \ell_\Gamma(\xi) + \wp_\Gamma(\xi)$$

for each $\xi \in \Gamma$. The antipode, κ, admits the unique extension $\hat{\kappa} : \Gamma^\wedge \to \Gamma^\wedge$, which is graded-antimultiplicative and satisfies $\hat{\kappa}\mathrm{d} = \mathrm{d}\hat{\kappa}$.

Let us denote by $\Gamma_{\mathrm{inv}}^\otimes$ and $\Gamma_{\mathrm{inv}}^\wedge$ subalgebras of left-invariant elements of $\Gamma^\otimes$ and $\Gamma^\wedge$ respectively. We have

$$\Gamma_{\mathrm{inv}}^\otimes = \sum_{k\geqslant 0}^{\oplus} \Gamma_{\mathrm{inv}}^{\otimes k} \qquad \Gamma_{\mathrm{inv}}^\wedge = \sum_{k\geqslant 0}^{\oplus} \Gamma_{\mathrm{inv}}^{\wedge k}$$

where $\Gamma_{\rm inv}^{\otimes k}$ and $\Gamma_{\rm inv}^{\wedge k}$ consist of left-invariant elements from $\Gamma^{\otimes k}$ and $\Gamma^{\wedge k}$ respectively. The space $\Gamma_{\rm inv}^{\otimes k}$ is actually the tensor product of k-copies of $\Gamma_{\rm inv}$.

The following natural isomorphism holds

$$\Gamma_{\rm inv}^{\wedge} = \Gamma_{\rm inv}^{\otimes}/S_{\rm inv}^{\wedge}$$

where $S_{\rm inv}^{\wedge}$ is the left-invariant part of $S^{\wedge}$. This space is an ideal in $\Gamma_{\rm inv}^{\otimes}$ generated by elements of the form

$$q = \pi(a^{(1)}) \otimes \pi(a^{(2)})$$

where $a \in \mathcal{R}$.

All introduced spaces of the form $\Gamma_{\rm inv}^{*}$ are right-invariant. We shall denote by ϖ^{*} the adjoint actions of G on the corresponding spaces.

The formula

$$\vartheta \circ a = \kappa(a^{(1)})\vartheta a^{(2)}$$

defines an extension of the right $\mathcal{A}$-module structure $\circ$ from $\Gamma_{\rm inv}$ to $\Gamma_{\rm inv}^{\wedge,\otimes}$. We have

$$1 \circ a = \epsilon(a)1$$
$$(\vartheta\eta) \circ a = (\vartheta \circ a^{(1)})(\eta \circ a^{(2)})$$

for each $\vartheta, \eta \in \Gamma_{\rm inv}^{\wedge,\otimes}$ and $a \in \mathcal{A}$.

The algebra $\Gamma_{\rm inv}^{\wedge} \subseteq \Gamma^{\wedge}$ is d-invariant. The differential $\mathrm{d} : \Gamma_{\rm inv}^{\wedge} \rightarrow \Gamma_{\rm inv}^{\wedge}$ is explicitly determined by

$$\mathrm{d}\pi(a) = -\pi(a^{(1)})\pi(a^{(2)}).$$

If Γ is *-covariant then the *-involution $* : \Gamma \rightarrow \Gamma$ is naturally extendible from Γ to $\Gamma^{\wedge,\otimes}$ (such that for each $\vartheta, \eta \in \Gamma^{\wedge,\otimes}$ we have $(\vartheta\eta)^* = (-)^{\partial\vartheta\partial\eta}\eta^*\vartheta^*$). Algebras $\Gamma_{\rm inv}^{\wedge}, \Gamma_{\rm inv}^{\otimes} \subseteq \Gamma^{\wedge,\otimes}$ are *-invariant. We have

$$(\vartheta \circ a)^* = \vartheta^* \circ \kappa(a)^*$$

for each $a \in \mathcal{A}$ and $\vartheta \in \Gamma_{\rm inv}^{\wedge,\otimes}$.

Explicitly, the *-involution on $\Gamma_{\rm inv}$ is determined by

$$\pi(a)^* = -\pi[\kappa(a)^*].$$

The map $\hat{\phi}$, as well as the left and the right actions of G on $\Gamma^{\wedge,\otimes}$ are *-preserving, in a natural manner.

Let M be a compact smooth manifold. By definition [8] a *quantum principal G-bundle* over M is a triplet $P = (\mathcal{B}, \mathrm{i}, F)$ where $\mathcal{B}$ is a (unital) *-algebra, consisting of appropriate 'functions' on P, while $F : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ and $\mathrm{i} : S(M) \rightarrow \mathcal{B}$ are (unital) *-homomorphisms, intrepretable as the dualized right action of G on P, and the dualized projection of P on M. Further, the bundle P is *locally trivial* in the sense that for each $x \in M$ there exists an open set $U \subseteq M$ such that $x \in U$, and a *-homomorphism $\pi_U : \mathcal{B} \rightarrow S(U) \otimes \mathcal{A}$ such that

$$\pi_U \mathrm{i}(f) = (f\restriction_U) \otimes 1$$
$$\pi_U(\mathcal{B}) \supseteq S_c(U) \otimes \mathcal{A}$$
$$(\mathrm{id} \otimes \phi)\pi_U = (\pi_U \otimes \mathrm{id})F$$

and such that

$$\pi_U(\mathrm{i}(f)b) = 0 \Longrightarrow \mathrm{i}(f)b = 0$$

for each $f \in S_c(U)$. Here S and S_c denote the corresponding *-algebras of complex smooth functions (with compact supports, respectively).

The homomorphism π_U is interpretable as the dualized trivialization of P over U. Every pair (U, π_U), consisting of an open set $U \subseteq M$ and of a *-homomorphism $\pi_U : \mathcal{B} \to S(U) \otimes \mathcal{A}$, satisfying above conditions is called a local trivialization of P.

A trivialization system for P is a family $\tau = \{(U, \pi_U) | U \in \mathcal{U}\}$ of local trivializations of P, where $\mathcal{U}$ is a finite open cover of M.

For each $k \in \mathbb{N}$ we shall denote by $N^k(\mathcal{U})$ the set of k-tuples $(U_1, \ldots, U_k) \in \mathcal{U}^k$ such that $U_1 \cap \cdots \cap U_k \neq \emptyset$.

The main structural result concerning quantum principal bundles is that there exists a natural correspondence between quantum principal G-bundles, P, and classical principal G_{cl}-bundles, P_{cl}, over M. This corresponence can be described as follows.

From a given trivialization system, τ, it is possible to construct the corresponding G-cocycle which is a system of *-automorphisms ψ_{UV} of $S(U \cap V) \otimes \mathcal{A}$, where $(U, V) \in N^2(\mathcal{U})$, realizing transformations between (V, π_V) and (U, π_U). Such systems of maps completely determine the bundle P.

Explicitly, let us consider the *-algebra

$$\Sigma(\mathcal{U}) = \sum_{U \in \mathcal{U}}^{\oplus} [S(U) \otimes \mathcal{A}].$$

The algebra $\mathcal{B}$ is realizable as a subalgebra of $\Sigma(\mathcal{U})$, consisting of elements $b \in \Sigma(\mathcal{U})$ satisfying

$$({}_U\!\restriction_{U \cap V} \otimes \mathrm{id}) p_U(b) = \psi_{UV} ({}_V\!\restriction_{U \cap V} \otimes \mathrm{id}) p_V(b)$$

for each $(U, V) \in N^2(\mathcal{U})$. Here, $p_U : \Sigma(\mathcal{U}) \to S(U) \otimes \mathcal{A}$ are coordinate projections. In terms of this realization we have

$$\pi_U = p_U \restriction \mathcal{B}$$

for each $U \in \mathcal{U}$.

However, it turns out that G-cocycles are in a natural bijection with standard G_{cl}-cocycles (over $\mathcal{U}$), which are systems of smooth maps $g_{UV} : U \cap V \to G_{\text{cl}}$ satisfying

$$g_{UV} g_{VW}(x) = g_{UW}(x)$$

for each $(U, V, W) \in N^3(\mathcal{U})$ and $x \in U \cap V \cap W$ (in particular $g_{UV}^{-1} = g_{VU}$). The correspondence is established via the following formula

$$\psi_{UV}(\varphi \otimes a) = \varphi g_{VU}(a^{(1)}) \otimes a^{(2)}.$$

Here, maps g_{UV} are understood as *-homomorphisms $g_{UV} : \mathcal{A} \to S(U)$, in a natural manner. On the other hand, G_{cl}-cocycles determine, in the standard manner, principal G_{cl}-bundles, P, over M.

The bundle P_{cl} is interpretable as the 'classical part' of P. The elements of P_{cl} are in a natural bijection with *-characters of $\mathcal{B}$. The correspondence $P \leftrightarrow P_{\text{cl}}$ has a simple geometrical explanation. The 'transition functions' ψ_{UV} are, at the geometrical level, vertical 'diffeomorphisms' of $(U \cap V) \times G$. Therefore they preserve the geometrical structure of $(U \cap V) \times G$. In particular, they must preserve the classical part $(U \cap V) \times G_{\text{cl}}$ consisting of points of $(U \cap V) \times G$. Moreover, transition diffeomorphisms are completely determined by their restrictions on $(U \cap V) \times G_{\text{cl}}$, because of the right covariance. The corresponding 'restrictions' are precisely transition functions for the classical bundle P_{cl}.

We pass to the study of differential calculus. For each (nonempty) open set $U \subseteq M$ let $\Omega(U)$ be the graded-differential *-algebra of differential forms on U. In developing a differential calculus over quantum principal bundles it is natural to assume that the calculus is fully compatible with the geometrical structure on the bundle, such that all local trivializations of the bundle also locally trivialize the calculus (a precise formulation of this condition is given in [8, section 3]). It turns out that this condition completely fixes the calculus on the bundle (if the calculus on the structure quantum group is fixed). However, the condition implies certain restrictions on a possible differential calculus, Γ, over G.

Namely, all retrivialization maps, ψ_{UV}, must be extendible to differential algebra automorphisms $\psi_{UV}^{\wedge} : \Omega(U \cap V)\hat{\otimes}\Gamma^{\wedge} \to \Omega(U \cap V)\hat{\otimes}\Gamma^{\wedge}$. Differential calculi, Γ, satisfying this condition are called admissible. If Γ is left-covariant then it is admissible iff

$$(X \otimes \mathrm{id})\,\mathrm{ad}(\mathcal{R}) = \{0\}$$

for each $X \in \mathrm{lie}(G_{\mathrm{cl}})$. This fact implies that there exists the minimal admissible left-covariant calculus Γ. This calculus is based on the right $\mathcal{A}$-ideal $\hat{\mathcal{R}} \subseteq \ker(\epsilon)$ consisting of all elements $a \in \ker(\epsilon)$ satisfying

$$(X \otimes \mathrm{id})\,\mathrm{ad}(a) = 0$$

for each $X \in \mathrm{lie}(G_{\mathrm{cl}})$.

Moreover, we have $\mathrm{ad}(\hat{\mathcal{R}}) \subseteq \hat{\mathcal{R}} \otimes \mathcal{A}$ and $\kappa(\hat{\mathcal{R}})^* = \hat{\mathcal{R}}$, which implies [27] that Γ is bicovariant and *-covariant respectively.

In the following, Γ will be this minimal admissible (bicovariant *-)calculus. Let $\Omega(P)$ be the graded-differential *-algebra representing differential calculus on P (constructed by combining differential forms on M with the universal envelope $\Gamma^{\wedge}$ of Γ). Explicitly, let us consider the direct sum

$$\Sigma^{\wedge}(\mathcal{U}) = \sum_{U \in \mathcal{U}}^{\oplus} [\Omega(U)\hat{\otimes}\Gamma^{\wedge}].$$

Then $\Omega(P)$ can be viewed as a graded-differential subalgebra consisting of elements $w \in \Sigma^{\wedge}(\mathcal{U})$ satisfying

$$({}_U{\restriction}_{U\cap V} \otimes \mathrm{id})\,p_U(w) = \psi_{UV}^{\wedge}({}_V{\restriction}_{U\cap V} \otimes \mathrm{id})\,p_V(w)$$

for each $(U, V) \in N^2(\mathcal{U})$. Here $p_U : \Sigma^{\wedge}(\mathcal{U}) \to \Omega(U)\hat{\otimes}\Gamma^{\wedge}$ are corresponding coordinate projections.

As a differential algebra, $\Omega(P)$ is generated by $\mathcal{B} = \Omega^0(P)$. For every local trivialization (U, π_U) of P there exists the unique differential algebra homomorphism $\pi_U^{\wedge} : \Omega(P) \to \Omega(U)\hat{\otimes}\Gamma^{\wedge}$ extending π_U (in fact $\pi_U^{\wedge} = p_U{\restriction}\Omega(P)$). The map $\mathrm{i} : S(M) \to \mathcal{B}$ admits a natural extension $\mathrm{i}^{\wedge} : \Omega(M) \to \Omega(P)$, which is interpretable as the 'pull back' of differential forms on M to P. We have

$$\pi_U^{\wedge}\mathrm{i}^{\wedge}(w) = (w{\restriction}_U) \otimes 1.$$

The right action $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is (uniquely) extendible to a differential algebra homomorphism $\hat{F} : \Omega(P) \to \Omega(P)\hat{\otimes}\Gamma^{\wedge}$, imitating the corresponding pull-back map. The formula

$$F^{\wedge} = (\mathrm{id} \otimes \Pi)\hat{F}$$

determines a *-homomorphism $F^{\wedge} : \Omega(P) \to \Omega(P) \otimes \mathcal{A}$ interpretable as the (dualized) right action of G on $\Omega(P)$. Here $\Pi : \Gamma^{\wedge} \to \mathcal{A}$ is the projection map.

Let $\mathfrak{ver}(P)$ be the graded-differential *-algebra obtained by factorizing $\Omega(P)$ through the (differential *-ideal) generated by elements of the form $\mathrm{d}[\mathrm{i}(f)]$. The elements of $\mathfrak{ver}(P)$ play the role of 'verticalized' differential forms on P (in classical geometry, entities obtained by restricting the domain of differential forms to the Lie algebra of vertical vector fields on the bundle). At the level of graded vector spaces, there exists a natural isomorphism

$$\mathfrak{ver}(P) \cong \mathcal{B} \otimes \Gamma_{\mathrm{inv}}^{\wedge}.$$

Let $\pi_v : \Omega(P) \to \mathfrak{ver}(P)$ be the corresponding projection map. In terms of the above identifications, the differential *-algebra structure on $\mathfrak{ver}(P)$ is specified by

$$(q \otimes \eta)(b \otimes \vartheta) = \sum_k q b_k \otimes (\eta \circ a_k)\vartheta$$

$$(b \otimes \vartheta)^* = \sum_k b_k^* \otimes (\vartheta^* \circ a_k^*)$$

$$d_v(b \otimes \vartheta) = \sum_k b_k \otimes \pi(a_k)\vartheta + b \otimes \mathrm{d}\vartheta$$

where $F(b) = \sum_k b_k \otimes a_k$.

Another important algebra naturally associated to $\Omega(P)$ is a graded *-subalgebra $\mathfrak{hor}(P) \subseteq \Omega(P)$ representing horizontal forms. By definition, $\mathfrak{hor}(P)$ consists of forms $w \in \Omega(P)$ with the property

$$\pi_U^{\wedge}(w) \in \Omega(U) \otimes \mathcal{A}$$

for each local trivialization (U, π_U). Equivalently,

$$\mathfrak{hor}(P) = (\hat{F})^{-1}\{\Omega(P) \otimes \mathcal{A}\}.$$

The algebra $\mathfrak{hor}(P)$ is invariant under the right action of G, in other words

$$F^{\wedge}(\mathfrak{hor}(P)) \subseteq \mathfrak{hor}(P) \otimes \mathcal{A}.$$

Let $\psi(P)$ be the space of all linear maps $\varphi : \Gamma_{\mathrm{inv}} \to \Omega(P)$ satisfying

$$(\varphi \otimes \mathrm{id})\varpi = F^{\wedge}\varphi.$$

This space is naturally graded (the grading is induced from $\Omega(P)$). The elements of $\psi(P)$ are quantum counterparts of pseudotensorial forms on the bundle with coefficients in the structure group Lie algebra (relative to the adjoint representation). The space $\psi(P)$ is closed with respect to compositions with $\mathrm{d} : \Omega(P) \to \Omega(P)$.

Let $\tau(P) \subseteq \psi(P)$ be the subspace consisting of $\mathfrak{hor}(P)$-valued maps. This space is imaginable as consisting of the corresponding tensorial forms.

There exists a natural *-involution on $\psi(P)$. It is given by

$$\varphi^*(\vartheta) = \varphi(\vartheta^*)^*.$$

The space $\tau(P)$ is *-invariant.

Tensorial forms possess the following local representation:

$$\pi_U^{\wedge}\varphi(\vartheta) = (f^U \otimes \mathrm{id})\varpi(\vartheta)$$

where $f^U : \Gamma_{\mathrm{inv}} \to \Omega(U)$ is a linear map.

For the purposes of this paper the most important topic of the theory of quantum principal bundles is the formalism of connections. By definition, a connection on P is every pseudotensorial Hermitian one-form ω satisfying

$$\pi_v\omega(\vartheta) = 1 \otimes \vartheta$$

for each $\vartheta \in \Gamma_{\mathrm{inv}}$. The above formula is the quantum counterpart for the classical condition that connections map fundamental vector fields into their generators. Connections form a real affine space $\mathfrak{con}(P)$.

In local terms, connections possess the following representation

$$\pi_U^\wedge \omega(\vartheta) = (A^U \otimes \mathrm{id})\varpi(\vartheta) + 1_U \otimes \vartheta$$

where $A^U : \Gamma_{\mathrm{inv}} \to \Omega(U)$ is a one-form valued Hermitian linear map (playing the role of the corresponding gauge potential).

The curvature operator can be described as follows. Let us fix a map $\delta : \Gamma_{\mathrm{inv}} \to \Gamma_{\mathrm{inv}} \otimes \Gamma_{\mathrm{inv}}$ which intertwines the corresponding adjoint actions and such that if

$$\delta(\vartheta) = \sum_k \vartheta_k^1 \otimes \vartheta_k^2$$

then

$$\delta(\vartheta^*) = -\sum_k (\vartheta_k^2)^* \otimes (\vartheta_k^1)^* \qquad \mathrm{d}\vartheta = \sum_k \vartheta_k^1 \vartheta_k^2.$$

Every such map will be called an *embedded differential*. Further, for each pair of linear maps φ, ψ on Γ_{inv} with values in an arbitrary algebra Ω let $\langle \varphi, \psi \rangle : \Gamma_{\mathrm{inv}} \to \Omega$ be a map given by

$$\langle \varphi, \psi \rangle(\vartheta) = \sum_k \varphi(\vartheta_k^1)\psi(\vartheta_k^2).$$

By construction, if $\varphi, \psi \in \psi(P)$ then also $\langle \varphi, \psi \rangle \in \psi(P)$.

Finally, the curvature R_ω of a connection ω can be defined as

$$R_\omega = \mathrm{d}\omega - \langle \omega, \omega \rangle.$$

The above formula corresponds to the structure equation in the classical theory. It turns out that R_ω is a tensorial two-form. Locally, in terms of the corresponding gauge potentials we have

$$\pi_U^\wedge R_\omega(\vartheta) = (F^U \otimes \mathrm{id})\varpi(\vartheta)$$

where

$$F^U = \mathrm{d}A^U - \langle A^U, A^U \rangle.$$

For each open set, $U \subseteq M$, the symbol $\otimes_U$ will be used for the tensor product over $S(U)$. Similarly, the symbol $\hat{\otimes}_U$ will denote the graded tensor product of graded-differential *-algebras containing $\Omega(U)$ as their subalgebra.

3. Quantum gauge bundles

This section is devoted to generalizations of the most important aspects of the concept of gauge transformations, in the framework of the formalism of quantum principal bundles. The main geometrical object that will be constructed is the quantum gauge bundle, a noncommutative-geometric counterpart of the gauge bundle of the classical theory.

3.1. Classical picture

In order to present motivations for constructions of this section let us assume for a moment that G is an ordinary compact Lie group, and let P be a (classical) principal bundle over M.

By definition, gauge transformations of P are vertical automorphisms of this bundle. In other words, gauge transformations are diffeomorphisms $\psi : P \to P$ satisfying

$$\pi_M\psi = \pi_M$$
$$\psi(pg) = \psi(p)g$$

for each $p \in P$ and $g \in G$, where $(p, g) \mapsto pg$ is the right action of G on P and $\pi_M : P \to M$ is the projection map. Equivalently, gauge transformations are interpretable as (smooth) sections of the gauge bundle $\mathcal{G}(P)$, which is the bundle associated to P, with respect to the adjoint action of G onto itself.

The equivalence between two definitions is established via the following formula

$$\psi(p) = pf(p)$$

where $f : P \to G$ is a smooth equivariant function in the sense that

$$f(pg) = g^{-1}f(p)g$$

for each $p \in P$ and $g \in G$. Such functions are in natural correspondence with sections of the corresponding associated bundle $\mathcal{G}(P)$.

For each $x \in M$ the fibre $G_x = \pi_M^{\sharp-1}(x)$ over x (where $\pi_M^\sharp : \mathcal{G}(P) \to M$ is the projection map) possesses a natural Lie group structure. The group G_x is isomorphic (generally noninvariantly) to G. For a given $p \in \pi_M^{-1}(x) = P_x$ there exists a canonical diffeomorphism $G \leftrightarrow P_x$ defined by $g \leftrightarrow pg$, and a group isomorphism $G \leftrightarrow G_x$ given by $g \leftrightarrow [(p, g)]$. Here, $\mathcal{G}(P)$ is understood as the orbit space of the right action $((p, g'), g) \mapsto (pg, g^{-1}g'g)$ of G on $P \times G$ and [ ] denotes the corresponding orbit.

There exists a natural left action of G_x on P_x. In terms of the above identifications this action becomes the multiplication on the left. Collecting all these fibre actions together, we obtain a smooth map

$$\beta_M^* : \mathcal{G}(P) \times_M P \to P. \tag{3.1}$$

With the help of β_M^* the equivalence between gauge transformations ψ and sections $\varphi : M \to \mathcal{G}(P)$ can be described as follows

$$\psi = \beta_M^*(\varphi \times_M \mathrm{id}). \tag{3.2}$$

Moreover, the correspondence $\psi \leftrightarrow \varphi$ is an isomorphism between the group $\mathcal{G}$ of gauge transformations of P, and the group $\Gamma(\mathcal{G}(P))$ of smooth sections of $\mathcal{G}(P)$.

The group structure in fibres of $\mathcal{G}(P)$ determine the following maps of bundles

$$\begin{aligned} &\text{the fibrewise multiplication} \quad \phi_M^* : \mathcal{G}(P) \times_M \mathcal{G}(P) \to \mathcal{G}(P)\\ &\text{the unit section} \quad \epsilon_M^* : M \to \mathcal{G}(P)\\ &\text{the fibrewise inverse} \quad \kappa_M^* : \mathcal{G}(P) \to \mathcal{G}(P). \end{aligned} \tag{3.3}$$

At the dual level of function algebras (3.1) and (3.3) are represented by the corresponding $S(M)$-linear *-homomorphisms

$$\begin{aligned} \phi_M &: S(\mathcal{G}(P)) \to S(\mathcal{G}(P)) \otimes_M S(\mathcal{G}(P))\\ \epsilon_M &: S(\mathcal{G}(P)) \to S(M)\\ \kappa_M &: S(\mathcal{G}(P)) \to S(\mathcal{G}(P))\\ \beta_M &: S(P) \to S(\mathcal{G}(P)) \otimes_M S(\mathcal{G}(P)). \end{aligned} \tag{3.4}$$

The structure of the gauge group is completely encoded in maps $\{\phi_M, \kappa_M, \epsilon_M\}$.

At the dual level, gauge transformations, ψ, can be viewed as $S(M)$-linear *-automorphisms $\psi : S(P) \to S(P)$ intertwining the (dualized) right action of G. Further, interpreted as sections of $\mathcal{G}(P)$, gauge transformations become, at the dual level, $S(M)$-linear *-homomorphisms $\varphi : S(\mathcal{G}(P)) \to S(M)$. In this picture, the action of $\mathcal{G}$ on $S(\mathcal{G}(P))$ is given by

$$(\varphi, f) \mapsto (\varphi \otimes \mathrm{id})\beta_M(f).$$

Maps (3.4) are not suitable for considering situations in which gauge transformations act on differential forms. This can be easily 'improved' by extending these maps to $\Omega(M)$-linear homomorphisms

$$\begin{aligned}
\hat{\phi}_M &: \Omega(\mathcal{G}(P)) \to \Omega(\mathcal{G}(P))\hat{\otimes}_M\Omega(\mathcal{G}(P)) \\
\hat{\epsilon}_M &: \Omega(\mathcal{G}(P)) \to \Omega(M) \\
\hat{\kappa}_M &: \Omega(\mathcal{G}(P)) \to \Omega(\mathcal{G}(P)) \\
\hat{\beta}_M &: \Omega(P) \to \Omega(\mathcal{G}(P))\hat{\otimes}_M\Omega(\mathcal{G}(P))
\end{aligned} \tag{3.5}$$

of graded-differential *-algebras. It is worth noting that the above maps are unique, as graded-differential extensions. Actually, these maps can be viewed as 'pull backs' of (3.1) and (3.3).

3.2. Quantum consideration

The presented picture admits a direct noncommutative-geometric generalization. First, we shall construct, starting from a quantum principal bundle P, the corresponding quantum gauge bundle $\mathcal{G}(P)$. Then the counterparts of maps (3.4) will be introduced and analysed. In analogy with the classical case we shall define gauge transformations as vertical automorphisms of the bundle P. It turns out that such gauge transformations of P are in a natural bijection with ordinary gauge transformations of the classical part, P_{cl}, of P. We shall also study various equivalent interpretations of gauge transformations. Finally, a canonical differential calculus on the bundle $\mathcal{G}(P)$ will be constructed and analysed.

Let G be a compact matrix quantum group, and let $P = (\mathcal{B}, \mathrm{i}, F)$ be a quantum principal G-bundle over M. Let us fix a trivialization system, τ, for P. For each $(U, V) \in N^2(\mathcal{U})$ let us define a linear map $\xi_{UV} : S(U \cap V) \otimes \mathcal{A} \to S(U \cap V) \otimes \mathcal{A}$ by the following formula

$$\xi_{UV}(\varphi \otimes a) = \varphi g_{UV}[\kappa(a^{(1)})a^{(3)}] \otimes a^{(2)}. \tag{3.6}$$

Lemma 3.1.

(i) The maps ξ_{UV} are $S(U \cap V)$-linear *-automorphisms and

$$\xi_{UV}^{-1} = \xi_{VU}. \tag{3.7}$$

(ii) We have

$$\xi_{UV}\xi_{VW}(\varphi) = \xi_{UW}(\varphi) \tag{3.8}$$

for each $(U, V, W) \in N^3(\mathcal{U})$ and $\varphi \in S_c(U \cap V \cap W) \otimes \mathcal{A}$.

(iii) The diagrams

$$\begin{array}{ccc}
S(U \cap V) \otimes \mathcal{A} & \xrightarrow{\mathrm{id}\otimes\phi} & [S(U \cap V) \otimes \mathcal{A}] \otimes_{U\cap V} [S(U \cap V) \otimes \mathcal{A}] \\
\downarrow{\scriptstyle \xi_{UV}} & & \downarrow{\scriptstyle \xi_{UV}\otimes\xi_{UV}} \\
S(U \cap V) \otimes \mathcal{A} & \xrightarrow[\mathrm{id}\otimes\phi]{} & [S(U \cap V) \otimes \mathcal{A}] \otimes_{U\cap V} [S(U \cap V) \otimes \mathcal{A}]
\end{array}$$

$$
\begin{array}{ccc}
S(U\cap V)\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\epsilon} & S(U\cap V)\\
{\scriptstyle\xi_{UV}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{id}}\\
S(U\cap V)\otimes\mathcal{A} & \xrightarrow[\mathrm{id}\otimes\epsilon]{} & S(U\cap V)
\end{array}
\qquad
\begin{array}{ccc}
S(U\cap V)\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\kappa} & S(U\cap V)\otimes\mathcal{A}\\
{\scriptstyle\xi_{UV}}\big\downarrow & & \big\downarrow{\scriptstyle\xi_{UV}}\\
S(U\cap V)\otimes\mathcal{A} & \xrightarrow[\mathrm{id}\otimes\kappa]{} & S(U\cap V)\otimes\mathcal{A}
\end{array}
$$

$$
\begin{array}{ccc}
S(U\cap V)\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\phi} & [S(U\cap V)\otimes\mathcal{A}]\otimes_{U\cap V}[S(U\cap V)\otimes\mathcal{A}]\\
{\scriptstyle\psi_{UV}}\big\downarrow & & \big\downarrow{\scriptstyle\xi_{UV}\otimes\psi_{UV}}\\
S(U\cap V)\otimes\mathcal{A} & \xrightarrow[\mathrm{id}\otimes\phi]{} & [S(U\cap V)\otimes\mathcal{A}]\otimes_{U\cap V}[S(U\cap V)\otimes\mathcal{A}]
\end{array}
$$

are commutative.

Proof. We have

$$
\begin{aligned}
\xi_{UV}\xi_{VW}(\varphi\otimes a) &= \xi_{UV}(\varphi g_{VW}[\kappa(a^{(1)})a^{(3)}]\otimes a^{(2)})\\
&= \varphi g_{VW}[\kappa(a^{(1)})a^{(5)}]g_{UV}[\kappa(a^{(2)})a^{(4)}]\otimes a^{(3)}\\
&= \varphi g_{UW}[\kappa(a^{(1)})a^{(3)}]\otimes a^{(2)} = \xi_{UW}(\varphi\otimes a)
\end{aligned}
$$

for each $(U,V,W)\in N^3(\mathcal{U})$, $\varphi\in S_c(U\cap V\cap W)$ and $a\in\mathcal{A}$. In particular, for $W=V$ this implies that the maps ξ_{UV} are bijective and that (3.7) holds.

The maps ξ_{UV} are *-homomorphisms because of

$$
\begin{aligned}
\xi_{UV}(\varphi^*\otimes a^*) &= \varphi^* g_{VU}(a^{(1)*})g_{UV}(a^{(3)*})\otimes a^{(2)*}\\
&= [\varphi g_{VU}(a^{(1)})g_{UV}(a^{(3)})]^*\otimes a^{(2)*} = \xi_{UV}(\varphi\otimes a)^*
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_{UV}(\varphi\psi\otimes ab) &= \varphi\psi g_{VU}(a^{(1)}b^{(1)})g_{UV}(a^{(3)}b^{(3)})\otimes a^{(2)}b^{(2)}\\
&= [\varphi g_{VU}(a^{(1)})g_{UV}(a^{(3)})\otimes a^{(2)}][\psi g_{VU}(b^{(1)})g_{UV}(b^{(3)})\otimes b^{(2)}]\\
&= \xi_{UV}(\varphi\otimes a)\xi_{UV}(\psi\otimes b).
\end{aligned}
$$

Finally, let us check the commutativity of the above diagrams. We compute

$$
\begin{aligned}
(\xi_{UV}\otimes\xi_{UV})(\mathrm{id}\otimes\phi)(\varphi\otimes a) &= \varphi g_{UV}(\kappa(a^{(1)})a^{(3)}\kappa(a^{(4)})a^{(6)})\otimes a^{(2)}\otimes a^{(5)}\\
&= \varphi g_{UV}(\kappa(a^{(1)})a^{(4)})\otimes a^{(2)}\otimes a^{(3)}\\
&= (\mathrm{id}\otimes\phi)\xi_{UV}(\varphi\otimes a)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
(\xi_{UV}\otimes\psi_{UV})(\mathrm{id}\otimes\phi)(\varphi\otimes a) &= \varphi g_{UV}(\kappa(a^{(1)})a^{(3)})g_{VU}(a^{(4)})\otimes a^{(2)}\otimes a^{(5)}\\
&= \varphi g_{VU}(a^{(1)})\otimes a^{(2)}\otimes a^{(3)} = (\mathrm{id}\otimes\phi)\psi_{UV}(\varphi\otimes a).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\xi_{UV}(\varphi\otimes\kappa(a)) &= \varphi g_{UV}(\kappa^2(a^{(3)})\kappa(a^{(1)}))\otimes\kappa(a^{(2)})\\
&= \varphi g_{UV}(\kappa(a^{(1)})a^{(3)})\otimes\kappa(a^{(2)})\\
&= (\mathrm{id}\otimes\kappa)\xi_{UV}(\varphi\otimes a).
\end{aligned}
$$

Together with a trivial observation that

$$
(\mathrm{id}\otimes\epsilon)\xi_{UV}(\varphi\otimes a) = \varphi g_{UV}(\kappa(a^{(1)})a^{(2)}) = \epsilon(a)\varphi
$$

this completes the proof. □

The (algebra of functions on the) quantum gauge bundle $\mathcal{G}(P)$ can be now constructed as follows. Let $\mathcal{D}$ be the set of elements $q \in \Sigma(\mathcal{U})$ such that

$$({}_U{\restriction}_{U\cap V} \otimes \mathrm{id})p_U(q) = \xi_{UV}({}_V{\restriction}_{U\cap V} \otimes \mathrm{id})p_V(q) \tag{3.9}$$

for each $(U, V) \in N^2(\mathcal{U})$.

Clearly, $\mathcal{D}$ is a *-subalgebra of $\Sigma(\mathcal{U})$. The quantum space $\mathcal{G}(P)$ corresponding to $\mathcal{D}$ plays the role of the bundle associated to the principal bundle P, with respect to the adoint action of G onto itself (represented by $\mathrm{ad} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$). The fact that $\mathcal{G}(P)$ is a bundle over M is established through the existence of a *-monomorphism $j_M : S(M) \to \mathcal{D}$, playing the role of the dualized fibring of $\mathcal{G}(P)$ over M. This map is defined by equalities

$$p_U j_M(f) = (f{\restriction}U) \otimes 1. \tag{3.10}$$

Definition 3.1. The pair $\mathcal{G}(P) = (\mathcal{D}, j_M)$ is called *the quantum gauge bundle* associated to P.

We are going to introduce quantum counterparts of maps ϕ_M, κ_M, ϵ_M and β_M. For each $U \in \mathcal{U}$, let $\pi_U^\sharp : \mathcal{D} \to S(U) \otimes \mathcal{A}$ be the restriction of p_U on $\mathcal{D}$.

Proposition 3.2.

(i) There exist the unique linear maps $\phi_M : \mathcal{D} \to \mathcal{D} \otimes_M \mathcal{D}$, $\epsilon_M : \mathcal{D} \to S(M)$, $\kappa_M : \mathcal{D} \to \mathcal{D}$ and $\beta_M : \mathcal{B} \to \mathcal{D} \otimes_M \mathcal{B}$ such that

$$(\pi_U^\sharp \otimes \pi_U^\sharp)\phi_M = (\mathrm{id} \otimes \phi)\pi_U^\sharp \tag{3.11}$$

$$(\pi_U^\sharp \otimes \pi_U)\beta_M = (\mathrm{id} \otimes \phi)\pi_U^\sharp \tag{3.12}$$

$$\pi_U^\sharp \kappa_M = (\mathrm{id} \otimes \kappa)\pi_U^\sharp \tag{3.13}$$

$${\restriction}_U \epsilon_M = (\mathrm{id} \otimes \epsilon)\pi_U^\sharp \tag{3.14}$$

for each $U \in \mathcal{U}$. Here, $S(U) \otimes \mathcal{A} \otimes \mathcal{A}$ and $(S(U) \otimes \mathcal{A}) \otimes_U (S(U) \otimes \mathcal{A})$ are identified, in a natural manner.

(ii) All maps are $S(M)$-linear. The maps ϕ_M ,ϵ_M and β_M are *-homomorphisms while κ_M is antimultiplicative and

$$\kappa_M[\kappa_M(f^*)^*] = f \tag{3.15}$$

for each $f \in \mathcal{D}$.

Proof. The above equalities uniquely fix the values of maps ϕ_M, ϵ_M, κ_M and β_M because the maps π_U and $\pi_U^\sharp$ distinguish points of $\mathcal{B}$ and $\mathcal{D}$.

Let us consider the algebra

$$\Sigma^*(\mathcal{U}) = \sum_{U \in \mathcal{U}}^{\oplus} S(U) \otimes \mathcal{A} \otimes \mathcal{A}.$$

Algebras $\mathcal{D} \otimes_M \mathcal{D}$ and $\mathcal{D} \otimes_M \mathcal{B}$ are understandably subalgebras of $\Sigma^*(\mathcal{U})$. Let us consider maps $\phi_M : \Sigma(\mathcal{U}) \to \Sigma^*(\mathcal{U})$, $\kappa_M : \Sigma(\mathcal{U}) \to \Sigma(\mathcal{U})$ and $\epsilon_M : \Sigma(\mathcal{U}) \to S(\mathcal{U})$ defined by

$$p_U^* \phi_M = (\mathrm{id} \otimes \phi)p_U$$

$$p_U \kappa_M = (\mathrm{id} \otimes \kappa)p_U$$

$${\restriction}_U \epsilon_M = (\mathrm{id} \otimes \epsilon)p_U$$

where $p_U^* : \Sigma^*(\mathcal{U}) \to S(U) \otimes \mathcal{A} \otimes \mathcal{A}$ are coordinate projections and $S(\mathcal{U})$ is the direct sum of algebras $S(U)$.

It is easy to see that $\phi_M(\mathcal{B}) \subseteq \mathcal{D} \otimes_M \mathcal{B}$, $\phi_M(\mathcal{D}) \subseteq \mathcal{D} \otimes_M \mathcal{D}$, $\kappa_M(\mathcal{D}) \subseteq \mathcal{D}$ and $\epsilon_M(\mathcal{D}) \subseteq S(M)$. Let us denote by $\{\phi_M, \beta_M, \kappa_M, \epsilon_M\}$ the corresponding restrictions. By construction (3.11)–(3.14) hold, maps β_M ,ϕ_M and ϵ_M are *-homomorphisms, κ_M is antimultiplicative and (3.15) holds. □

The fibres of the bundle $\mathcal{G}(P)$ possess a natural quantum group structure. Further, the bundle $\mathcal{G}(P)$ acts on the bundle P, preserving fibres and the right action. This is a geometrical background for the next proposition.

Proposition 3.3. The following identities hold

$$(\mathrm{id}\otimes\phi_M)\phi_M = (\phi_M\otimes\mathrm{id})\phi_M \tag{3.16}$$
$$(\mathrm{id}\otimes F)\beta_M = (\beta_M\otimes\mathrm{id})F \tag{3.17}$$
$$(\mathrm{id}\otimes\beta_M)\beta_M = (\phi_M\otimes\mathrm{id})\beta_M \tag{3.18}$$
$$(\mathrm{id}\otimes\epsilon_M)\phi_M = (\epsilon_M\otimes\mathrm{id})\phi_M = \mathrm{id} \tag{3.19}$$
$$(\epsilon_M\otimes\mathrm{id})\beta_M = \mathrm{id} \tag{3.20}$$
$$m_M(\kappa_M\otimes\mathrm{id})\phi_M = m_M(\mathrm{id}\otimes\kappa_M)\phi_M = j_M\epsilon_M \tag{3.21}$$

where $m_M : \mathcal{D}\otimes_M\mathcal{D}\to\mathcal{D}$ is the multiplication map.

Proof. In terms of local trivializations, everything reduces to elementary algebraic properties of the coproduct, the co-unit, and the antipode. □

We pass to the analysis of gauge transformations, in this quantum framework. In analogy with classical geometry, these transformations will be defined as vertical automorphisms of the bundle.

Definition 3.2. A gauge transformation of the bundle P is every $S(M)$-linear *-automorphism $\gamma : \mathcal{B}\to\mathcal{B}$ such that the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{B}\otimes\mathcal{A} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma\otimes\mathrm{id}} \\ \mathcal{B} & \xrightarrow[F]{} & \mathcal{B}\otimes\mathcal{A} \end{array} \tag{3.22}$$

is commutative.

The above diagram infers that γ intertwines the right action of G on P, while the $S(M)$-linearity property ensures that γ is a 'vertical' automorphism of P. Obviously, gauge transformations form a subgroup $\mathcal{G}\subseteq\mathrm{Aut}(\mathcal{B})$.

Proposition 3.4.

(i) The formula

$$f\leftrightarrow(f\otimes\mathrm{id})\beta_M = \gamma \tag{3.23}$$

establishes a bijection between $S(M)$-linear *-homomorphisms $f : \mathcal{D}\to S(M)$ and gauge transformations $\gamma\in\mathcal{G}$. In terms of this correspondence, the map ϵ_M corresponds to the neutral element in $\mathcal{G}$ while the product and the inverse in the gauge group are given by

$$f\kappa_M\leftrightarrow\gamma^{-1} \tag{3.24}$$
$$(f'\otimes f)\phi_M\leftrightarrow\gamma\gamma'. \tag{3.25}$$

(ii) Let γ be an arbitrary gauge transformation. Then the map $\gamma_{\mathrm{cl}} : P_{\mathrm{cl}}\to P_{\mathrm{cl}}$ defined by

$$\gamma_{\mathrm{cl}}(p) = p\gamma^{-1} \tag{3.26}$$

is an ordinary gauge transformation of P_{cl}. Moreover, the above formula establishes an isomorphism between groups of gauge transformations of bundles P and P_{cl}.

Proof. Identity (3.17) implies that a $S(M)$-linear homomorphism $\gamma : \mathcal{B} \to \mathcal{B}$ given by the right-hand side of (3.23) satisfies (3.22). Identity (3.20) ensures that ϵ_M corresponds to the neutral element of $\mathcal{G}$.

Let us consider an arbitrary gauge transformation $\gamma \in \mathcal{G}$. In terms of the trivialization system, τ, we have

$$\pi_U \gamma(b) = \sum_i \varphi_i \gamma_U(a_i^{(1)}) \otimes a_i^{(2)} \tag{3.27}$$

for each $U \in \mathcal{U}$. Here, $\pi_U(b) = \sum_i \varphi_i \otimes a_i$ and $\gamma_U : U \to G_{\mathrm{cl}}$ are smooth functions uniquely determined by γ (understood here in the 'dual' manner). We have

$$(\gamma_V(a^{(1)}){\restriction}_{U\cap V}) g_{VU}(a^{(2)}) = g_{VU}(a^{(1)})(\gamma_U(a^{(2)})\ {\restriction}_{U\cap V}) \tag{3.28}$$

for each $a \in \mathcal{A}$ and $(U, V) \in N^2(\mathcal{U})$.

Conversely, if *-homomorphisms $\gamma_U : \mathcal{A} \to S(U)$ are given such that equalities (3.28) hold, then formula (3.27) consistently determines a gauge transformation γ.

Let us now consider a map $f : \Sigma(\mathcal{U}) \to S(\mathcal{U})$ defined by

$$f = \sum_{U\in\mathcal{U}}^{\oplus} f_U$$

where $f_U : S(U) \otimes \mathcal{A} \to S(U)$ are maps given by $f_U(\varphi \otimes a) = \varphi\gamma_U(a)$. It is easy to see that if $b \in \mathcal{D}$ then $f(b) \in S(M)$ (where $S(M)$ is understood as a subalgebra of $S(\mathcal{U})$). Let us pass to the corresponding restriction $f : \mathcal{D} \to S(M)$. By construction (3.23) holds (it is evident in a local trivialization). Conversely, if $f : \mathcal{D} \to S(M)$ determines a gauge transformation γ then

$$f(b){\restriction}_U = \sum_i \varphi_i \gamma_U(a_i).$$

This easily follows from (3.23).

Let us check correspondences (3.24), (3.25). We have

$$[(f \otimes f')\phi_M \otimes \mathrm{id}]\beta_M = (f \otimes \gamma')\beta_M = \gamma'\gamma$$
$$(f\kappa_M \otimes f)\phi_M = f m_M(\kappa_M \otimes \mathrm{id})\phi_M = \epsilon_M.$$

Finally, the second statement easily follows from the definition of gauge transformations, and from the local expression (3.27) for them. □

A geometrical explanation of statement (ii) is the following. Gauge transformations, being diffeomorphisms of P at the geometrical level, must preserve classical and quantum parts of P. On the other hand, because of the intertwining property, gauge transformations, γ, are completely determined by their 'restrictions' γ_{cl} on P_{cl}, which correspond precisely to the standard gauge transformations of P_{cl}.

The quantum gauge bundle $\mathcal{G}(P)$ is also an inherently inhomogeneous geometrical object. This is a consequence of the inhomogeneity of G. The classical part of the bundle $\mathcal{G}(P)$ (*-characters on $\mathcal{D}$) is naturally identificable with the ordinary gauge bundle of P_{cl}. In other words,

$$(\mathcal{G}(P))_{\mathrm{cl}} = \mathcal{G}(P_{\mathrm{cl}}).$$

Let $f : \mathcal{D} \to S(M)$ be the *-epimorphism corresponding to $\gamma \in \mathcal{G}$. This map determines a section f^* of the bundle $(\mathcal{G}(P))_{\mathrm{cl}}$ as follows

$$[f^*(x)](\varphi) = [f(\varphi)](x) \tag{3.29}$$

where $x \in M$ and $\varphi \in \mathcal{D}$. In the framework of correspondence (3.23), the map f^* becomes the section corresponding to the gauge transformation γ_{cl}, in the classical manner.

We pass to the construction and the study of differential calculus on the bundle $\mathcal{G}(P)$. The calculus will be constructed by combining differential forms on the base manifold M with a differential calculus on the quantum group G. This calculus will be based on the universal differential envelope $\Gamma^\wedge$ of the minimal admissible first-order bicovariant *-calculus Γ over G.

Lemma 3.5.

(i) For each $(U, V) \in N^2(\mathcal{U})$ there exists the unique homomorphism $\xi^\wedge_{UV} : \Omega(U \cap V)\hat{\otimes}\Gamma^\wedge \to \Omega(U \cap V)\hat{\otimes}\Gamma^\wedge$ of (graded) differential algebras, extending the map ξ_{UV}. The map $\xi^\wedge_{UV}$ is *-preserving and bijective, and

$$(\xi^\wedge_{UV})^{-1} = \xi^\wedge_{VU}. \tag{3.30}$$

(ii) We have

$$\xi^\wedge_{UV}\xi^\wedge_{VW}(\varphi) = \xi^\wedge_{UW}(\varphi) \tag{3.31}$$

for each $(U, V, W) \in N^3(\mathcal{U})$ and $\varphi \in \Omega_c(U \cap V \cap W)\hat{\otimes}\Gamma^\wedge$.

(iii) The diagrams

$$\begin{array}{ccc}
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow{\mathrm{id}\otimes\hat{\phi}} & [\Omega(U\cap V)\otimes\Gamma^\wedge]\hat{\otimes}_{U\cap V}[\Omega(U\cap V)\otimes\Gamma^\wedge] \\
\downarrow{\scriptstyle \xi^\wedge_{UV}} & & \downarrow{\scriptstyle \xi^\wedge_{UV}\otimes\xi^\wedge_{UV}} \\
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow[\mathrm{id}\otimes\hat{\phi}]{} & [\Omega(U\cap V)\otimes\Gamma^\wedge]\hat{\otimes}_{U\cap V}[\Omega(U\cap V)\otimes\Gamma^\wedge]
\end{array}$$

$$\begin{array}{ccc}
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow{\mathrm{id}\otimes\epsilon\Pi} & \Omega(U\cap V) \\
\downarrow{\scriptstyle \xi^\wedge_{UV}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow[\mathrm{id}\otimes\epsilon\Pi]{} & \Omega(U\cap V)
\end{array}
\qquad
\begin{array}{ccc}
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow{\mathrm{id}\otimes\hat{\kappa}} & \Omega(U\cap V)\hat{\otimes}\Gamma^\wedge \\
\downarrow{\scriptstyle \xi^\wedge_{UV}} & & \downarrow{\scriptstyle \xi^\wedge_{UV}} \\
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow[\mathrm{id}\otimes\hat{\kappa}]{} & \Omega(U\cap V)\hat{\otimes}\Gamma^\wedge
\end{array}$$

$$\begin{array}{ccc}
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow{\mathrm{id}\otimes\hat{\phi}} & [\Omega(U\cap V)\otimes\Gamma^\wedge]\hat{\otimes}_{U\cap V}[\Omega(U\cap V)\otimes\Gamma^\wedge] \\
\downarrow{\scriptstyle \psi^\wedge_{UV}} & & \downarrow{\scriptstyle \xi^\wedge_{UV}\otimes\psi^\wedge_{UV}} \\
\Omega(U\cap V)\hat{\otimes}\Gamma^\wedge & \xrightarrow[\mathrm{id}\otimes\hat{\phi}]{} & [\Omega(U\cap V)\otimes\Gamma^\wedge]\hat{\otimes}_{U\cap V}[\Omega(U\cap V)\otimes\Gamma^\wedge]
\end{array}$$

are commutative.

Proof. The uniqueness of $\xi^\wedge_{UV}$ follows from the fact that $\Omega_c(U \cap V)\hat{\otimes}\Gamma^\wedge$ is generated, as a differential algebra, by $S_c(U \cap V) \otimes \mathcal{A}$. The hermicity of $\xi^\wedge_{UV}$ follows from the fact that $*\xi^\wedge_{UV}*$ is a differential extension of the same map $*\xi_{UV}* = \xi_{UV}$. In a similar way it follows from lemma 3.1 that the above diagrams are commutative, and that (3.30), (3.31) hold.

We prove the existence of $\xi^\wedge_{UV}$. The admissibility of Γ and the universality of $\Gamma^\wedge$ imply that maps g_{UV} admit the unique graded-differential (*-preserving) extensions $\hat{g}_{UV} : \Gamma^\wedge \to \Omega(U \cap V)$.

Now, the maps $f_{UV} : \Gamma^\wedge \to \Omega(U \cap V)\hat{\otimes}\Gamma^\wedge$ given by

$$f_{UV}(w) = \sum_i (\hat{g}_{VU}(w_i^1) \otimes w_i^2)(\hat{g}_{UV}(w_i^3) \otimes 1)$$

where $\sum_i w_i^1 \otimes w_i^2 \otimes w_i^3 = (\hat{\phi}\otimes\mathrm{id})\hat{\phi}(w) = (\mathrm{id}\otimes\hat{\phi})\hat{\phi}(w)$, are homomorphisms of differential *-algebras. Finally, let $\xi^\wedge_{UV}$ be defined by

$$\xi^\wedge_{UV}(\alpha \otimes w) = \alpha f_{UV}(w).$$

It is evident that such defined maps are differential algebra homomorphisms extending ξ_{UV}. □

Let $\Omega(\tau, \mathcal{G}(P))$ be the set of all elements $w \in \Sigma^\wedge(\mathcal{U})$ satisfying

$$({}_U\!\restriction_{U\cap V} \otimes \mathrm{id})p_U(w) = \xi^\wedge_{UV}({}_V\!\restriction_{U\cap V} \otimes \mathrm{id})p_V(w) \tag{3.32}$$

for each $(U, V) \in N^2(\mathcal{U})$. It is clear that $\Omega(\tau, \mathcal{G}(P))$ is a graded-differential *-subalgebra of $\Sigma^\wedge(\mathcal{U})$, and that $\Omega^0(\tau, \mathcal{G}(P)) = \mathcal{D}$. The elements of the algebra $\Omega(\tau, \mathcal{G}(P))$ play the role of differential forms on the bundle $\mathcal{G}(P)$. This algebra is generated by $\mathcal{D}$, and in fact does not depend of a trivialization system τ. More precisely, if η is another trivialization system for P then there exists (the unique) differential $(*-)$ isomorphism $\Omega(\tau, \mathcal{G}(P)) \leftrightarrow \Omega(\eta, \mathcal{G}(P))$ extending the identity map on $\mathcal{D}$. For this reason we shall simply write $\Omega(\tau, \mathcal{G}(P)) = \Omega(\mathcal{G}(P))$.

Proposition 3.6.

(i) The maps $\{\epsilon_M, j_M, \beta_M, \phi_M\}$ admit unique extensions

$$\phi^\wedge_M : \Omega(\mathcal{G}(P)) \to \Omega(\mathcal{G}(P))\hat{\otimes}_M\Omega(\mathcal{G}(P))$$
$$\epsilon^\wedge_M : \Omega(\mathcal{G}(P)) \to \Omega(M)$$
$$j^\wedge_M : \Omega(M) \to \Omega(\mathcal{G}(P))$$
$$\beta^\wedge_M : \Omega(P) \to \Omega(\mathcal{G}(P))\hat{\otimes}_M\Omega(P)$$

which are homomorphisms of graded-differential algebras.

(ii) The map κ_M admits the unique extension $\kappa^\wedge_M : \Omega(\mathcal{G}(P)) \to \Omega(\mathcal{G}(P))$ which is graded-antimultiplicative and satisfies

$$\kappa^\wedge_M d = d\kappa^\wedge_M. \tag{3.33}$$

(iii) The following identities hold

$$(\phi^\wedge_M \otimes \mathrm{id})\phi^\wedge_M = (\mathrm{id} \otimes\phi^\wedge_M)\phi^\wedge_M \tag{3.34}$$
$$(\phi^\wedge_M \otimes \mathrm{id})\beta^\wedge_M = (\mathrm{id} \otimes\beta^\wedge_M)\beta^\wedge_M \tag{3.35}$$
$$(\mathrm{id} \otimes\hat{F})\beta^\wedge_M = (\beta^\wedge_M \otimes \mathrm{id})\hat{F} \tag{3.36}$$
$$(\mathrm{id} \otimes\epsilon^\wedge_M)\phi^\wedge_M = (\epsilon^\wedge_M \otimes \mathrm{id})\phi^\wedge_M = \mathrm{id} \tag{3.37}$$
$$(\epsilon^\wedge_M \otimes \mathrm{id})\beta_M = \mathrm{id} \tag{3.38}$$
$$m^\wedge_M(\kappa^\wedge_M \otimes \mathrm{id})\phi^\wedge_M = m^\wedge_M(\mathrm{id} \otimes\kappa^\wedge_M)\phi^\wedge_M = j^\wedge_M\epsilon^\wedge_M \tag{3.39}$$

where $m^\wedge_M$ is the multiplication map in $\Omega(\mathcal{G}(P))$.

(iv) We have

$$*\kappa^\wedge_M* = (\kappa^\wedge_M)^{-1} \tag{3.40}$$

while $\{\epsilon^\wedge_M, j^\wedge_M, \beta^\wedge_M, \phi^\wedge_M\}$ are *-preserving maps.

Proof. Using the (anti)multiplicativity, the intertwining differentials properties, and the fact that all considered differential algebras are generated by corresponding zeroth-order subalgebras, it is easy to see that extensions of all maps involved are, if they exist, unique. The same properties, together with proposition 3.3, imply that identities (3.34)–(3.39) hold. Statement (iv) follows from (ii) of proposition 3.2 in a similar way. Finally, existence of maps $\epsilon^\wedge_M$, $j^\wedge_M$, $\beta^\wedge_M$, $\phi^\wedge_M$ and $\kappa^\wedge_M$ can be established in a similar way as for maps ϵ_M, j_M, β_M, ϕ_M and κ_M. □

Every $\gamma \in \mathcal{G}$ understood as a *-homomorphism $f : \mathcal{D} \to S(M)$ is uniquely extendible to a $\Omega(M)$-linear *-homomorphism $f^\wedge : \Omega(\mathcal{G}(P)) \to \Omega(M)$ of graded-differential algebras.

The following correspondences hold

$$\gamma^{-1} \leftrightarrow f^\wedge \kappa_M^\wedge \tag{3.41}$$

$$\gamma\gamma' \leftrightarrow m_M^\wedge (f^\wedge \otimes f'^\wedge)\phi_M^\wedge. \tag{3.42}$$

4. Gauge fields

In this section we shall present a generalization of the classical gauge theory, within the geometrical framework of quantum principal bundles. The base manifold M will play the role of spacetime. The quantum group G will describe 'internal symmetries'. In order to simplify considerations, we shall deal only with a 'pure gauge theory'.

Let us assume that $\Gamma_{\rm inv}$ is endowed with an ϖ-invariant scalar product $(\,,\,)$. This means that

$$(\vartheta, \eta) \otimes 1 = \sum_{kl} (\vartheta_k, \eta_l) \otimes c_k^* d_l$$

for each $\vartheta, \eta \in \Gamma_{\rm inv}$, where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$ and $\sum_l \eta_l \otimes d_l = \varpi(\eta)$.

Let us assume that M is oriented and endowed with a (pseudo)riemannian structure.

Let us denote by $\star$ the Hodge operation on $\Omega(M)$. It can be (uniquely) extended to a linear map $\star : \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ such that

$$\star(\mathrm{i}^\wedge(\alpha)b) = \mathrm{i}^\wedge(\star(\alpha))b$$

for each $\alpha \in \Omega(M)$ and $b \in \mathcal{B}$.

Following the classical analogy gauge fields will be geometrically represented by connection forms ω on the bundle P.

To make possible dynamical considerations it is necessary to fix a Lagrangian. Generalizing the classical situation, it is natural to consider Lagrangians which are quadratic functions of the curvature R_ω. The curvature operator R_ω depends, other than on the connection ω, also on a choice of the embedded differential map $\delta : \Gamma_{\rm inv} \to \Gamma_{\rm inv} \otimes \Gamma_{\rm inv}$. As a consequence of this, dynamical properties of the gauge theory will be essentially influenced by δ. In the classical case the curvature is δ-independent.

Let us consider a map $L : \mathfrak{con}(P) \to \mathfrak{hor}(P)$ given by

$$L(\omega) = \sum_i R_\omega(e_i)\star[R_\omega(\bar{e}_i)] \tag{4.1}$$

where elements e_i form an orthonormal system in $\Gamma_{\rm inv}$ and the bar denotes the conjugation in $\Gamma_{\rm inv}$. It is easy to see that $L(\omega)$ is independent of the choice of the mentioned orthonormal system.

The map L in fact takes values from the space $\Omega^n(M)$ (where n is the dimension of M). Indeed, in terms of local trivializations we have

$$\pi_U^\wedge[L(\omega)] = \sum_i F^U(e_i)\star[F^U(\bar{e}_i)] \otimes 1. \tag{4.2}$$

This easily follows from the fact that $\sum_i e_i \otimes \bar{e}_i$ is $\varpi^{\otimes 2}$-invariant.

We shall interpret the map L as the Lagrangian. In terms of the local representation, principal stationary points [14] of the corresponding action functional $S(\omega) = \int_M L(\omega)$ are given by the following equations of motion

$$d\star F^U(\bar{e}_k) - \tfrac{1}{2} \sum_{ij} (d_i^{jk} - d_i^{kj}) A^U(e_j)\star F^U(\bar{e}_i) = 0 \tag{4.3}$$

where numbers d_k^{ij} are determined by

$$\delta(e_k) = -\frac{1}{2} \sum_{ij} d_k^{ij} e_i \otimes e_j. \tag{4.4}$$

The above equations correspond to the classical Yang–Mills equations of motion. The numbers $(d_i^{jk} - d_i^{kj})/2$ play the role of the structure constants of (the Lie algebra of) G.

If the space Γ_{inv} is infinite-dimensional a technical difficulty arises, related to a question of convergence of the sum in (4.1), (4.2). In such cases, it is necessary to restrict possible values of ω on some subspace of $\mathfrak{con}(P)$, consisting of connections having sufficiently rapidly decreasing components, in an appropriate sense.

We pass to the study of symmetry properties of the introduced Lagrangian. At first, it is easy to see that $L(\omega)$ is invariant under gauge transformations of the bundle P.

The group $\mathcal{G}$ naturally acts on the left, via compositions, on the space $\psi(P)$ of pseudotensorial forms. The space $\tau(P)$ is invariant with respect to this action, because $\mathfrak{hor}(P)$ is $\mathcal{G}$ invariant. The connection space is also gauge invariant. In terms of gauge potentials the transformation of connections is

$$A^U(\vartheta) \longrightarrow \sum_k A^U(\vartheta_k)\gamma^U(c_k) + \partial^U(\vartheta). \tag{4.5}$$

Here, $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$, the map $\partial^U : \Gamma_{\mathrm{inv}} \to \Omega^1(U)$ is given by

$$\partial^U \pi(a) = \gamma^U \kappa(a^{(1)}) d\gamma^U(a^{(2)})$$

while $\gamma^U : \mathcal{A} \to S(U)$ is the map locally representing γ. Further, the transformation of the curvature is

$$F^U(\vartheta) \longrightarrow \sum_k F^U(\vartheta_k)\gamma^U(c_k). \tag{4.6}$$

The Lagrangian (4.1) is invariant under gauge transformations of the bundle P. This is a simple consequence of the unitarity of the representation ϖ.

This invariance is a manifestation of *classical symmetry properties* of the Lagrangian. These symmetry properties are completely expressible in terms of the classical part, P_{cl}, of P.

On the other hand, the Lagrangian $L(\omega)$ possesses symmetry properties which are not expressible in classical terms. The appearance of these ‘quantum symmetries’ is a purely quantum phenomenon caused by the quantum nature of the space G. Formally, they can be described as the invariance of the Lagrangian under a natural action of the quantum gauge bundle $\mathcal{G}(P)$.

Let $\psi(P, \mathcal{G}(P))$ be the space of linear maps $f : \Gamma_{\mathrm{inv}} \to \Omega(\mathcal{G}(P))\hat{\otimes}_M\Omega(P)$ satisfying

$$(f \otimes \mathrm{id})\varpi = (\mathrm{id} \otimes F^\wedge) f. \tag{4.7}$$

If $\varphi \in \psi(P)$ then $\beta_M^\wedge \varphi \in \psi(P, \mathcal{G}(P))$. Hence, it is possible to introduce the map $\beta_M^\wedge : \psi(P) \to \psi(P, \mathcal{G}(P))$ (via compositions).

Let us compute the element $\beta_M^\wedge \omega$ for $\omega \in \mathfrak{con}(P)$. Using the definition of $\beta_M^\wedge$ and the local expression for ω we obtain

$$(\pi_U^\sharp \otimes \pi_U)[\beta_M^\wedge(\omega)(\vartheta)] = 1_U \otimes 1 \otimes \vartheta + \sum_k \{A^U(\vartheta_k) \otimes c_k^{(1)} \otimes c_k^{(2)} + 1_U \otimes \vartheta_k \otimes c_k\}. \tag{4.8}$$

Here an identification

$$[\Omega(U)\hat{\otimes}\Gamma^\wedge]\hat{\otimes}_U[\Omega(U)\hat{\otimes}\Gamma^\wedge] = \Omega(U)\hat{\otimes}\Gamma^\wedge\hat{\otimes}\Gamma^\wedge$$

is assumed. It is worth noting that the transformation law (4.5) is contained in (4.8). Indeed, understanding gauge transformations as differential algebra homomorphisms $f^\wedge : \Omega(\mathcal{G}(P)) \to \Omega(M)$ we obtain (4.5) by composing $\beta_M^\wedge(\omega)$ and $f^\wedge \otimes \mathrm{id}$.

The curvature is transformed as follows

$$(\pi_U^\sharp \otimes \pi_U)[\beta_M^\wedge(R_\omega)(\vartheta)] = \sum_k F^U(\vartheta_k) \otimes c_k^{(1)} \otimes c_k^{(2)}. \tag{4.9}$$

That is, in local terms we have

$$F^U \longrightarrow (F^U \otimes \mathrm{id})\varpi. \tag{4.10}$$

The curvature operator is gauge covariant in the sense that

$$\beta_M^\wedge(R_\omega) = d\beta_M^\wedge(\omega) - \langle \beta_M^\wedge(\omega), \beta_M^\wedge(\omega)\rangle. \tag{4.11}$$

A possible interpretation of the above equation (which is a trivial consequence of the fact that $\beta_M^\wedge : \Omega(P) \to \Omega(\mathcal{G}(P))\hat{\otimes}_M \Omega(M)$ is a differential algebra homomorphism) is the following. The relation between the connection, ω, and its curvature is, being expressible in intrinsicly geometrical terms, preserved under the action of $\mathcal{G}(P)$. Expression (4.6) for the curvature of the transformed connection under a gauge transformation also directly follows from (4.9).

In order to find the transformation of the local expression for the Lagrangian, we should insert into (4.2) the local expression for the transformed curvature, under the action $\beta_M^\wedge$ of $\mathcal{G}(P)$ on P. The Lagrangian transforms as follows

$$\left\{\sum_k F^U(e_k)\star F^U(\bar{e}_k)\right\} \longrightarrow \sum_{kln} F^U(e_l)\star F^U(\bar{e}_n) \otimes c_{lk}c_{nk}^* \tag{4.12}$$

where $\varpi(e_i) = \sum_j e_j \otimes c_{ji}$. On the other hand

$$\left[\sum_k F^U(e_k)\star F^U(\bar{e}_k)\right] \otimes 1 = \sum_{kln} F^U(e_l)\star F^U(\bar{e}_n) \otimes c_{lk}c_{nk}^* \tag{4.13}$$

because of the $\varpi^{\otimes 2}$-invariance of $\sum_k e_k \otimes \bar{e}_k$.

Hence, the Lagrangian is invariant with respect to the action $\beta_M^\wedge$ of the gauge bundle $\mathcal{G}(P)$ on P.

It is important to mention that the property of 'quantum gauge invariance' of the Lagrangian cannot be viewed as an inherent property of the local expression (4.2). Because this property essentially depends on *the ordering of terms* F^U and $\star F^U$. However, in the general case, the ordering of terms R_ω and $\star R_\omega$ in the global representation of the Lagrangian is essential, because $\mathcal{B}$ is a noncommutative algebra.

5. An example

We shall now illustrate the presented formalism on a concrete example, assuming that $G = SU_\mu(2)$ (with $\mu = (-1, 1) \setminus \{0\}$). By definition [25] this compact matrix quantum group is based on the 2×2 matrix

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \tag{5.1}$$

where the elements α and γ satisfy the following relations

$$\begin{aligned} &\alpha\gamma = \mu\gamma\alpha \qquad \gamma\alpha^* = \mu\alpha^*\gamma \qquad \gamma\gamma^* = \gamma^*\gamma \\ &\alpha^*\alpha + \gamma^*\gamma = 1 \qquad \alpha\alpha^* + \mu^2\gamma^*\gamma = 1. \end{aligned} \tag{5.2}$$

The classical part of G is isomorphic to $U(1)$. An explicit isomorphism is given by $g \leftrightarrow g(\alpha)$.

It turns out [8, section 6] that the right $\mathcal{A}$-ideal $\hat{\mathcal{R}} \subseteq \ker(\epsilon)$ determining the minimal admissible (bicovariant $*-$) first-order calculus Γ over G is given by

$$\hat{\mathcal{R}} = (\mu^2\alpha + \alpha^* - (1+\mu^2)1)\ker(\epsilon). \tag{5.3}$$

Let $X : \mathcal{A} \to \mathbb{C}$ be a generator of $\mathrm{lie}(G_{\mathrm{cl}})$ specified by

$$\begin{aligned} X(\alpha) &= -X(\alpha^*) = \tfrac{1}{2} \\ X(\gamma) &= X(\gamma^*) = 0. \end{aligned} \tag{5.4}$$

Let $\rho : \mathcal{A} \to \mathcal{A}$ be a map given by $\rho = (X \otimes \mathrm{id})\,\mathrm{ad}$. Let $\nu : \Gamma_{\mathrm{inv}} \to \mathbb{C}$ and $\tilde{\rho} : \Gamma_{\mathrm{inv}} \to \mathcal{A}$ be the maps defined by $\nu\pi = X$ and $\tilde{\rho}\pi = \rho$. Then $\tilde{\rho} = (\nu \otimes \mathrm{id})\varpi$, and $\tilde{\rho}$ maps isomorphically the space Γ_{inv} onto the *-subalgebra $\mathcal{Q} \subseteq \mathcal{A}$ of left G_{cl}-invariant elements of $\mathcal{A}$. The subalgebra, $\mathcal{Q}$, is interpretable as the algebra of polynomial functions on a quantum two-sphere.

The adjoint action, ϖ, is reducible. The space Γ_{inv} is decomposable into the orthogonal sum

$$\Gamma_{\mathrm{inv}} = \sum_{k\geqslant 0}^{\oplus} \Gamma_{\mathrm{inv}}^k$$

of irreducible subspaces. The subspace Γ_{inv}^k is $(2k+1)$-dimensional (that is, all integer-spin irreducible multiplets are involved).

The space $\tilde{\rho}(\Gamma_{\mathrm{inv}}^k) = \mathcal{Q}_k$ is spanned by quantum spherical harmonics ζ_{km}, where $m \in \{-k, \ldots, k\}$. They constitute a standard basis for the action of G. Explicitly, these elements are given by

$$\begin{aligned} \zeta_{km} &= (-)^m \mu^{km-m}[(k-m)_\mu!/(k+m)_\mu!]^{1/2}\partial^m p_k(\gamma\gamma^*)\gamma^m\alpha^m \\ \zeta_{k,-m} &= \mu^{km}\alpha^{*m}\gamma^{*m}[(k-m)_\mu!/(k+m)_\mu!]^{1/2}\partial^m p_k(\gamma\gamma^*). \end{aligned} \tag{5.5}$$

Here, $m \geqslant 0$ and $\partial : P(x) \to P(x)$ is a 'quantum differential' (acting on the space $P(x)$ of x-polynoms) specified by $\partial(x^n) = n_\mu x^{n-1}$. Finally, $p_k(x)$ are polynomials given by

$$\begin{aligned} p_k(x) &= (-)^k c_k \partial^k \left[x^k \prod_{j=1}^{k} (1 - \mu^{2-2j}x) \right] \\ p_0(x) &= 1 \end{aligned} \tag{5.6}$$

while $c_k > 0$ and

$$k_\mu! = \prod_{j=1}^{k} j_\mu \qquad j_\mu = \frac{1-\mu^{2j}}{1-\mu^2}.$$

Let us now describe a construction of the natural embedded differential map δ. We shall first construct a complement $\mathcal{L} \subseteq \ker(\epsilon)$ of the space $\hat{\mathcal{R}}$.

The elements γ^k ($k \in \mathbb{N}$) are primitive for the adjoint action of G on $\ker(\epsilon)$. Let $\mathcal{L} \subseteq \ker(\epsilon)$ be the minimal ϖ-invariant subspace containing these elements, and the ad-invariant element $\mu^2\alpha + \alpha^* - (1+\mu^2)1$. It turns out that the restriction $(\pi\restriction\mathcal{L}) : \mathcal{L} \to \Gamma_{\mathrm{inv}}$ is bijective. Evidently, this restriction intertwines the adjoint actions. Let $\delta : \Gamma_{\mathrm{inv}} \to \Gamma_{\mathrm{inv}} \otimes \Gamma_{\mathrm{inv}}$ be defined by

$$\delta(\vartheta) = -(\pi \otimes \pi)\phi[(\pi\restriction\mathcal{L})^{-1}(\vartheta)]. \tag{5.7}$$

It is clear, by construction, that δ is an embedded differential map. Moreover,

$$\delta\varkappa = -(\varkappa \otimes \varkappa)\delta \tag{5.8}$$

where the extension of the antipode $\varkappa : \Gamma_{\mathrm{inv}} \to \Gamma_{\mathrm{inv}}$ is given by

$$\varkappa\pi = -\pi\kappa^2. \tag{5.9}$$

Let us compute the values of δ on the singlet and the triplet subspace of Γ_{inv}. The singlet space Γ^0_{inv} is spanned by

$$\varepsilon = \pi(\mu^2\alpha + \alpha^*) \tag{5.10}$$

while the triplet space Γ^1_{inv} is spanned by

$$\eta_+ = \pi(\gamma) \qquad \eta = \pi(\alpha - \alpha^*) \qquad \eta_- = \pi(\gamma^*). \tag{5.11}$$

Applying the definition of δ we obtain

$$-\delta(\varepsilon) = (\varepsilon \otimes \varepsilon + \mu^2\eta \otimes \eta)/(1+\mu^2) - \mu\eta_+ \otimes \eta_- - \mu^3\eta_- \otimes \eta_+$$
$$-\delta(\eta_+) = ((\varepsilon - \mu^2\eta) \otimes \eta_+ + \eta_+ \otimes (\varepsilon + \eta))/(1+\mu^2)$$
$$-\delta(\eta_-) = (\eta_- \otimes (\varepsilon - \mu^2\eta) + (\varepsilon + \eta) \otimes \eta_-)/(1+\mu^2)$$
$$-\delta(\eta) = (\varepsilon \otimes \eta + \eta \otimes \varepsilon + (1-\mu^2)\eta \otimes \eta)/(1+\mu^2) + \mu(\eta_+ \otimes \eta_- - \eta_- \otimes \eta_+).$$

The corresponding gauge theory based on the bundle P, calculus Γ group G and the Lagrangian $L(\omega)$ is essentially different from the classical gauge theory with $G = SU(2)$.

At first, gauge fields possess infinitely many internal degrees of freedom. In the classical limit $\mu \to 1$ the restriction $A^U \restriction \Gamma^1_{\mathrm{inv}}$ on the triplet subspace can be interpreted as a classical $SU(2)$ gauge field. Restrictions on other irreducible subspaces are classically interpretable as additional vector fields.

According to the general theory, the connection A^U can be decomposed into 'classical' and 'purely quantum' parts

$$A^U = A^U_{\mathrm{cl}} + A^U_{\perp}$$

where $A^U_{\mathrm{cl}} \restriction \ker(\nu) = 0$ and $A^U_{\perp}(\varepsilon) = 0$. The map A^U_{cl} can be interpreted as a connection on the classical $U(1)$-bundle P_{cl}. It is important to point out that the decomposition $\Gamma_{\mathrm{inv}} = \ker(\nu) {+} \mathbb{C}\varepsilon$ is incompatible with the decomposition of Γ_{inv} into irreducible multiplets.

Let us compute the singlet and the triplet components of the curvature. Applying the definition of δ and using the local expression of the curvature we find

$$F^U(\varepsilon) = \mathrm{d}A^U(\varepsilon) + \mu(1-\mu^2)A^U(\eta_-)A^U(\eta_+)$$
$$F^U(\eta_+) = \mathrm{d}A^U(\eta_+) + A^U(\eta_+)A^U(\eta)$$
$$F^U(\eta_-) = \mathrm{d}A^U(\eta_-) + A^U(\eta)A^U(\eta_-)$$
$$F^U(\eta) = \mathrm{d}A^U(\eta) + 2\mu A^U(\eta_+)A^U(\eta_-).$$

In general, components of the restriction $F^U \restriction \Gamma^k_{\mathrm{inv}}$ will be expressible through fields $A^U(\vartheta)$, where $\vartheta \in \Gamma^l_{\mathrm{inv}}$ and $1 \leqslant l \leqslant k$.

Equations of motion are mutually *essentially correlated*. Indeed, the equation describing the propagation of fields $A^U \restriction \Gamma^k_{\mathrm{inv}}$ will generally contain terms of the form $A^U(\vartheta) {\star} F^U(\eta)$, where $\vartheta, \eta \in \Gamma^{i,j}_{\mathrm{inv}}$ and $|i-k| \leqslant j \leqslant i+k$. This easily follows from the definition of δ. It is interesting to observe that nonsinglet components are not explicitly influenced by the singlet component $A^U(\varepsilon)$. On the other hand, the singlet propagation is intertwined only with $A^U(\eta_\pm)$. Explicitly,

$$d{\star}F^U(\varepsilon) = 0.$$

6. Concluding remarks

In this study we have presented a gauge theory over classical spacetime, in which internal symmetry groups are quantum. The basic elements of the formalism can be further naturally incorporated into the fully quantum context [14], where the spacetime is quantum. However, the admissibility of the calculus Γ is replaced by another global condition, since in the general quantum context it is not possible to speak in terms of local trivializations.

We have only considered the 'pure' gauge theory in this study. The matter fields can be introduced via the analogues of the associated vector bundles. In the general quantum context, M is represented by a noncommutative *-algebra $\mathcal{V} \leftrightarrow S(M)$, and it is natural to identify associated vector bundles with the intertwiner $\mathcal{V}$-bimodules $\mathcal{F}_u = \mathrm{Mor}(u, F)$ consisting of intertwining operators between finite-dimensional representations, u, of G and the action map F. It turns out that, quite generally [12], the system of all these bimodules completely determines the internal structure of the bundle.

In this study we have also assumed that the higher-order differential calculus on the structure group is based on the corresponding universal envelope. All constructions can be performed also in the case when the higher-order calculus is described by the corresponding bicovariant (braided) exterior algebra [27].

The admissibility assumption for Γ ensures full local trivializability of differential structures on P and $\mathcal{G}(P)$. However, from the 'local' point of view, the whole formalism works for an arbitrary bicovariant *-calculus Γ.

Physical properties of the presented gauge theory are essentially influenced by two additional structural elements. First, it is necessary to fix a bicovariant *-calculus Γ over G. This determines kinematical degrees of freedom. Secondly, the curvature is determined only after fixing an embedded differential map, δ, in such a way that the dynamics becomes δ-dependent. As an example of how the map δ can influence the dynamics, let us mention gauge fields based on a four-dimensional calculus [27] over quantum $SU(2)$. This (nonadmissible) calculus is spanned by the triplet $\{\eta_+, \eta, \eta_-\}$ and the singlet $\{\tau\}$. As explained in [8], changing appropriately δ we can pass from the model of noninteracting fields, to a model where the triplet fields interact similarly as the components of the classical $SU(2)$ gauge field, modulo the presence of the singlet τ.

A different quantum bundle formalism [2] was used in [3, 17] to construct a quantum analogue of standard gauge theory.

From our point of view, the geometrical formulation proposed in [2] lacks flexibility, because the basic entities of the formalism (covariant derivative, horizontal projection, curvature) can be constructed only in very special situations—for bundles possessing some additional properties, or using the universal calculus (where we have only trivial relations at the level of the calculus).

One possibility to overcome this difficulty is to consider only universal calculi, and to restrict the values of the curvature tensor to matrix elements of a given representation of the structure group. The embedded differential is then simulated by the action of the coproduct map on these matrix elements. However, this *a priori* excludes all nontrivial quantum phenomena that we have considered. Another interesting possibility for developing a gauge theory in the framework of [2] is given by quantum principal bundles possessing 'strong' connections [17], which gives an effective regularity condition. This works also for certain nonuniversal calculi on the bundle. Such connections are associated to the base, in the appropriate sense, and they can be taken as proper analogues of gauge fields.

The construction of quantum gauge bundles proposed in this paper depends on the classicality of the base space (and possibility to locally trivialize the bundle). For general quantum principal bundles, it is necessary to use essentially different methods [14]. General quantum gauge bundles can be constructed by combining the intrinsic braided structure on quantum principal bundles [15] with the natural structuralization [12] in terms of the intertwiner bimodules (this replaces, in a certain sense, local trivializations).

It is important to mention that, for general quantum principal bundles, it has been proven in [4] that gauge transformations (understood as vertical automorphisms of the bundle) are in a natural correspondence with the appropriate maps from the structure group algebra to the bundle algebra intertwining the adjoint action and the right action F. In particular, this property complements the correspondence established in proposition 3.4.

A conceptually different approach in constructing a quantum group gauge theory was proposed in [24], using the dual picture of quantized universal enveloping algebras. The gauge fields are based on the concept of the associated quantum Lie algebra [22].

In particular, the formulation proposed in [24] has a 'good' classical limit, because the dimension of the quantum Lie algebra is the same as the dimension of its classical counterpart. It would be very interesting to find an invariant geometrical formulation for such a structure.

From our point of view however, it is unnatural to expect the existence of such a classical limit, because of the explained inherent geometrical inhomogeneity of quantum groups. Furthermore, it is plausible to adopt the following interpretation.

In a gauge theory with a quantum group G, the 'true' local symmetries are described by the classical part G_{cl}. The 'complement' of G_{cl} in G is a 'purely quantum' space, describing 'deformed' symmetry-like properties. This residual symmetry should be able, in principle, to unify the particle multiplets associated to G_{cl}. Such an interpretation is very close to the supersymmetry philosophy. In accordance with this way of thinking, it is conceptually incorrect to try to *deform* the classical gauge theory. Instead, classical gauge theory should be *refined*, by considering an appropriate quantum extension of the classical internal symmetry group.

The concept of symmetry is logically independent of the concept of a quantum space. For example [13], it is possible to define consistently classical differential-geometric structures on a quantum space. A natural framework for such structures is given by quantum bundles possessing classical structure groups. If the structure group is classical, then the construction of the quantum gauge bundle is simplified, and can be completed [15] using the intrinsic (in this case involutive) braiding for quantum principal bundles.

Appendix. The minimal admissible calculus

In this appendix some properties of entities associated with the minimal admissible calculus, Γ, over the quantum $SU(2)$ group are collected. In particular, we shall analyse in more details the structure of the space $\mathcal{L}$ which determines the embedded differential map.

For each integer $n \geqslant 1$ let u_n be the $n \times n$ matrix over $\mathcal{A}$, corresponding to the irreducible representation [25, 26] of G, having the spin $(n-1)/2$ and acting in $\mathbb{C}^n$. Let $\mathcal{A}_n$ be the lineal spanned by matrix elements of u_n. We have

$$\mathcal{A} = \sum_{n \geqslant 1}^{\oplus} \mathcal{A}_n$$

according to the representation theory of G. The spaces $\mathcal{A}_n$ are invariant under the adjoint action of G. They are mutually orthogonal, relative to the scalar product induced by the Haar measure $h : \mathcal{A} \to \mathbb{C}$.

In subspaces $\mathcal{A}_n$ the adjoint action decomposes (without degeneracy) into irreducible multiplets with spins from the set $\{0, 1, \ldots, n-1\}$.

Lemma A.1. Let $\xi \in \mathcal{A}_n$ be a primitive element for the k-spin subrepresentation of $\mathrm{ad}\restriction\mathcal{A}_n$. Then,

$$\xi = p_{kn}(\lambda)\gamma^k \tag{A.1}$$

where $\lambda = \mu\alpha + \mu^{-1}\alpha^*$ and p_{kn} is a polynom of degree $n-k-1$ with real coefficients.

Proof. From the representation theory of G it follows that

$$\mathcal{A}_2\mathcal{A}_n = \mathcal{A}_n\mathcal{A}_2 = \mathcal{A}_{n-1} \oplus \mathcal{A}_{n+1}$$

for each $n \geqslant 2$. This implies that $\mathcal{A}_n \setminus \{0\}$ is consisting of certain polynoms of degree $n-1$ (over generators). Further, polynoms of degree $k \leqslant n-1$ form the space $\sum_i^{*\oplus} \mathcal{A}_i$, where $i \leqslant n$. Also, from the reality of commutation relations (5.2) and the orthogonality of spaces $\mathcal{A}_n$ it follows that we can write

$$\mathcal{A}_n = \mathcal{A}_n^{\Re} \oplus \mathrm{i}\mathcal{A}_n^{\Re}$$

where $\mathcal{A}_n^{\Re}$ consists of polynoms with real coefficients.

On the other hand, every nonzero element of the form (A.1) is primitive, and generates an irreducible k-spin multiplet relative to the adjoint representation. Keeping in mind the form of the decomposition of $\varpi\restriction\mathcal{A}_n$ into irreducible multiplets we conclude that (A.1) covers all primitive elements of the restriction $\varpi\restriction\mathcal{A}_n$. □

Let us assume that polynoms p_{kn} are fixed. For fixed k, polynoms p_{kn} are orthogonal, with respect to the scalar product given by

$$(p, q) = h(q(\lambda)\gamma^k\gamma^{*k}p(\lambda)^*). \tag{A.2}$$

Let $j : \mathcal{A} \to \mathcal{A}$ be the modular automorphism [26] corresponding to the Haar measure. This map is characterized by the identity

$$h(ba) = h(j(a)b). \tag{A.3}$$

In the case of the quantum $SU(2)$ group we have

$$\begin{aligned} j(\gamma) &= \gamma \qquad j(\alpha) = \mu^2\alpha \\ j(\alpha^*) &= \mu^{-2}\alpha^* \qquad j(\gamma^*) = \gamma^*. \end{aligned} \tag{A.4}$$

Applying (A.3), (A.4) we can see that the scalar product defined in (A.2) can be rewritten in the form

$$(p, q) = h[p^*(\lambda)q(\lambda)(\gamma\gamma^*)^k]. \tag{A.5}$$

Now, starting from (A.5), we observe that the above scalar product is invariant under the replacement $\lambda \to \alpha + \alpha^*$, and using elementary properties of polynoms it can be shown that all zeros of p_{kn} are contained in the interval $[-2, 2]$.

We have

$$\mathcal{L} = \mathbb{C}(\mu\lambda - (1+\mu^2)1) \oplus \left\{\sum_{k\geqslant 1}^{\oplus} \mathcal{L}_k\right\} \tag{A.6}$$

where $\mathcal{L}_k \subseteq \mathcal{A}_{k+1}$ is the k-spin irreducible subspace (for the adjoint action). Let $\mathcal{L}_* \subseteq \mathcal{A}$ be the lineal given by

$$\mathcal{L}_* = \mathbb{C}1 \oplus \left\{\sum_{k\geqslant 1}^{\oplus} \mathcal{L}_k\right\}. \tag{A.7}$$

Let $P(\lambda) \subseteq \mathcal{A}$ be the subalgebra generated by λ.

Lemma A.2.

(i) We have

$$\pi(\mathcal{A}_n) = \sum_{k\leqslant n-1}^{\oplus} \Gamma^k_{\text{inv}} \tag{A.8}$$

for each $n \geqslant 2$.

(ii) The map $\mho : P(\lambda) \otimes \mathcal{L}_* \to \mathcal{A}$ given by

$$\mho(p(\lambda) \otimes a) = p(\lambda)a \tag{A.9}$$

is bijective.

Proof. Let us prove that $\mho$ is bijective. First, let us observe that the elements from $P(\lambda)$ are ad-invariant. In particular,

$$\text{ad}(p(\lambda)a) = p(\lambda)\,\text{ad}(a) \tag{A.10}$$

for each $a \in \mathcal{A}$. According to lemma A.1 all primitive elements for $\text{ad} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ are contained in the image of $\mho$. Now (A.10) implies that $\mho$ is surjective. We prove that $\mho$ is injective. It is sufficient to check that $\mho\restriction(P(\lambda)\otimes\mathcal{L}_k)$ is injective, for each $k \in \mathbb{N}$. However, it follows again from lemma A.1 and (A.10), because $\mho(p_{kn}(\lambda) \otimes \mathcal{L}_k) \subseteq \mathcal{A}_n$ is exactly the k-spin irreducible subspace.

The following identity holds on non-trivial ad-multiplets

$$\rho\mho = (\epsilon \otimes (\rho\restriction\mathcal{L}_*)). \tag{A.11}$$

Statement (i) now follows from the definition of Γ_{inv} and from the facts that $\epsilon(\lambda) = \mu + \mu^{-1}$ and $p_{kn}(\mu + \mu^{-1}) \neq 0$. □

Using (A.6) and the definition of δ, it can be shown that

$$\delta(\Gamma^n_{\text{inv}}) \subseteq \sum_{ij\ *}^{\oplus} (\Gamma^i_{\text{inv}} \otimes \Gamma^j_{\text{inv}}) \tag{A.12}$$

for each $n \in \mathbb{N}$, where the sum is taken over pairs (i, j) satisfying $|i - j| \leqslant n \leqslant i + j$. In particular,

$$\delta(\vartheta)^{0,n} = d_n \varepsilon \otimes \vartheta$$
$$\delta(\vartheta)^{n,0} = d_n \vartheta \otimes \varepsilon$$

for each $\vartheta \in \Gamma^n_{\text{inv}}$, with $d_n \in \Re \setminus \{0\}$. This implies that singlet components of ω are not present in nonsinglet components of the curvature.

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